Séminaire LAL, Univ. Paris Sud - 22/04/14

Building gravitational waveforms from precessing black hole binary systems for advanced LIGO/Virgo

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arXiv 1308.3271

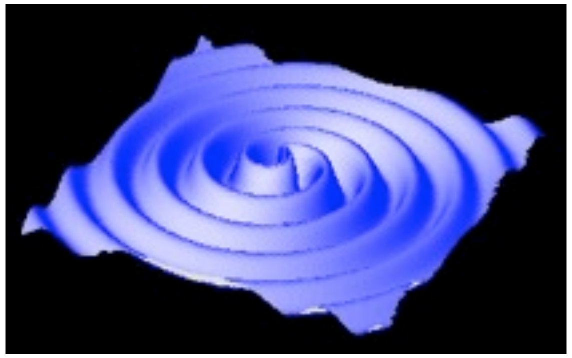
Outline

- Gravitational waves and Compact Binary Systems
- Tools for modeling Compact Binary Coalescences
- PhenomP: a closed form IMR model for precessing systems

Gravitational waves

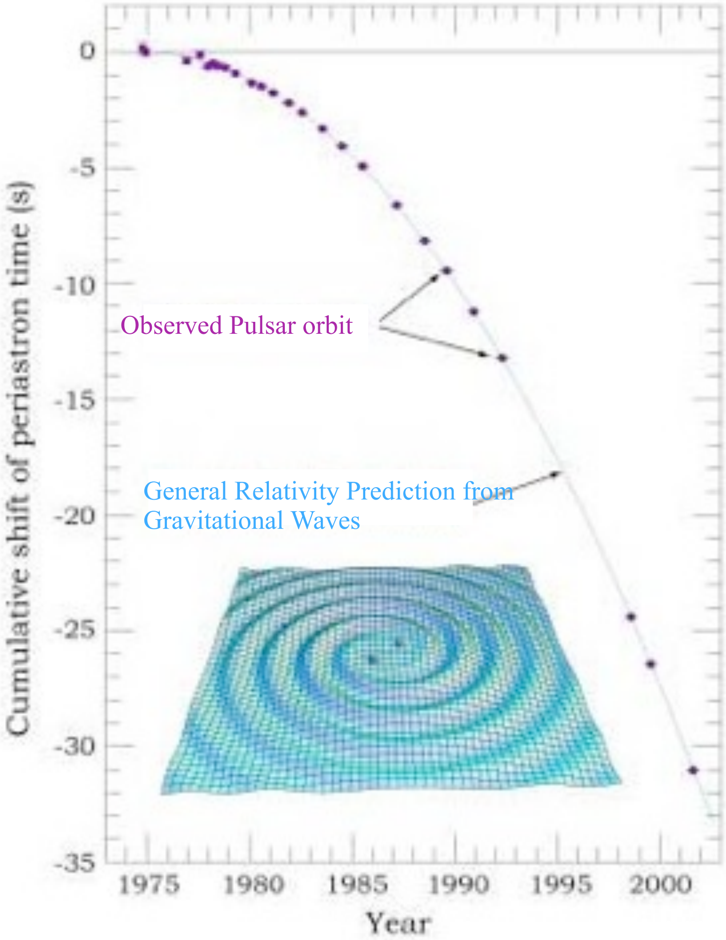
• Gravitational waves are one of the fundamental predictions of General Relativity in which gravity results from the curvature of space-time

• They are «ripples» in the fabric of space-time (perturbations of the metric) that propagate at the speed of light and that are generated by violent phenomena involving massive objects moving in a non axisymmetric way at relativistic speeds (more precisely, by systems with a time varying mass quadrupole or higher moment)



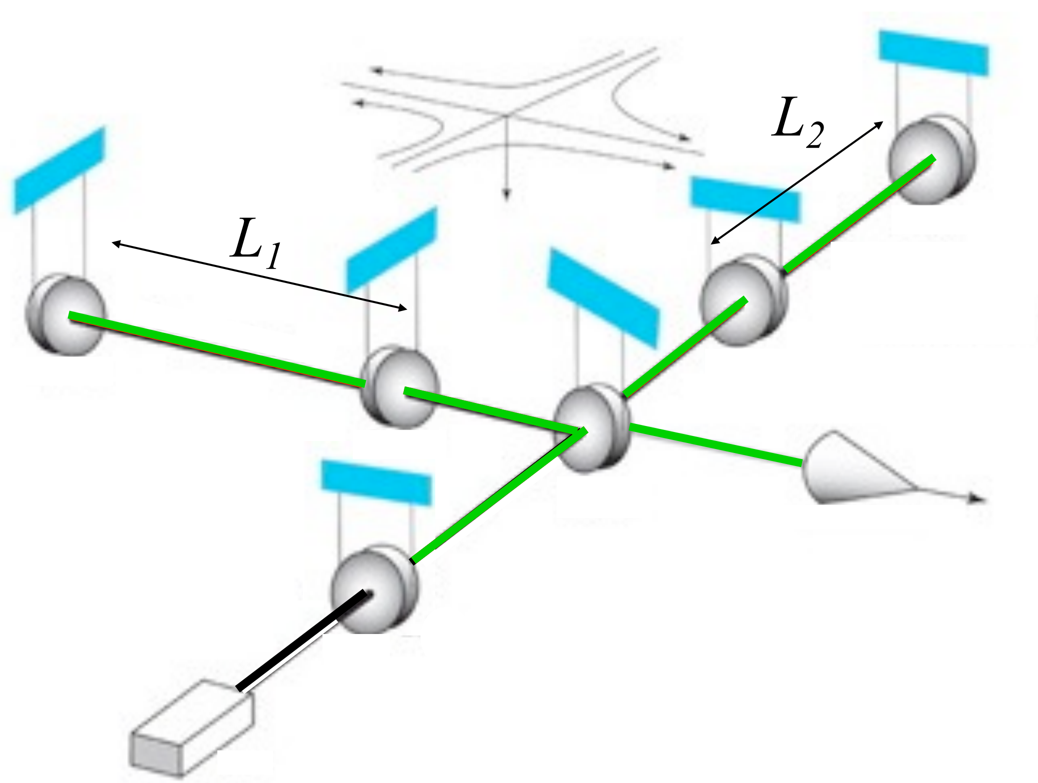
• Their very weak coupling to matter makes them very hard to detect. No direct detection so far but several indirect observations

- energy loss of binary pulsars (PSR B1913+16, Hulse & Taylor 1974, Nobel 93)
- BICEP2 !

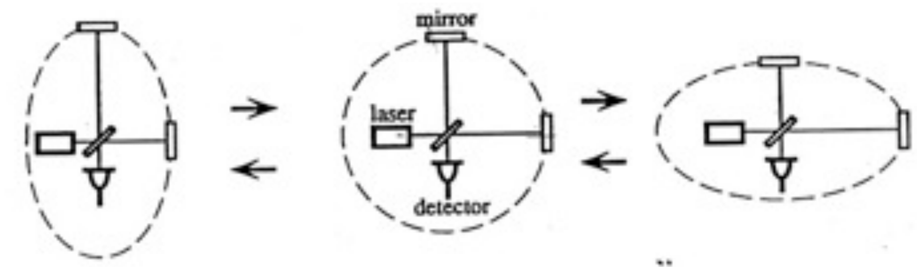


• Gravitational Waves will give a *non electromagnetic* view of the universe, and therefore opening a new window on the universe

Principle of detection with interferometers

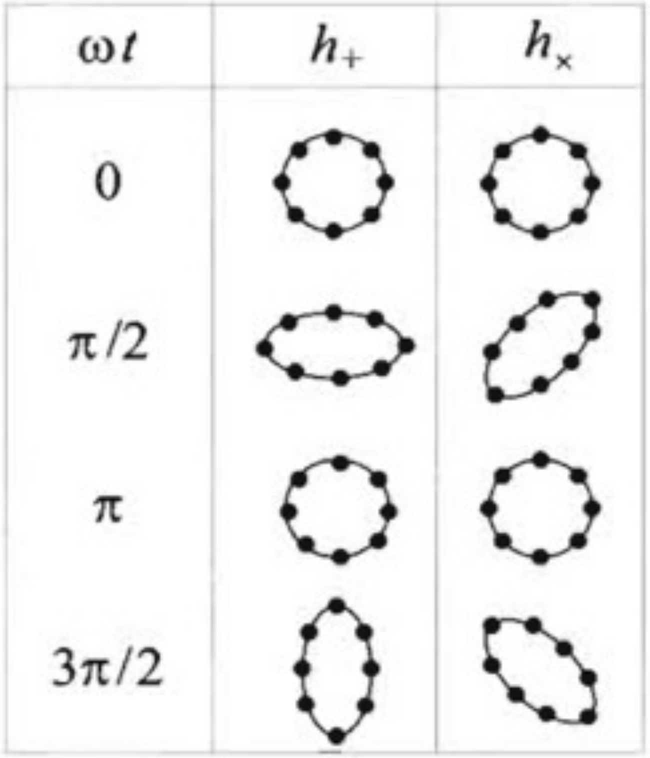


A GW passing through the detector changes the proper length of the arms of the interferometer



The strength of a gravitational wave is given by the strain h(t) = dL / L

Technological challenge: $h \approx 10^{-21}$



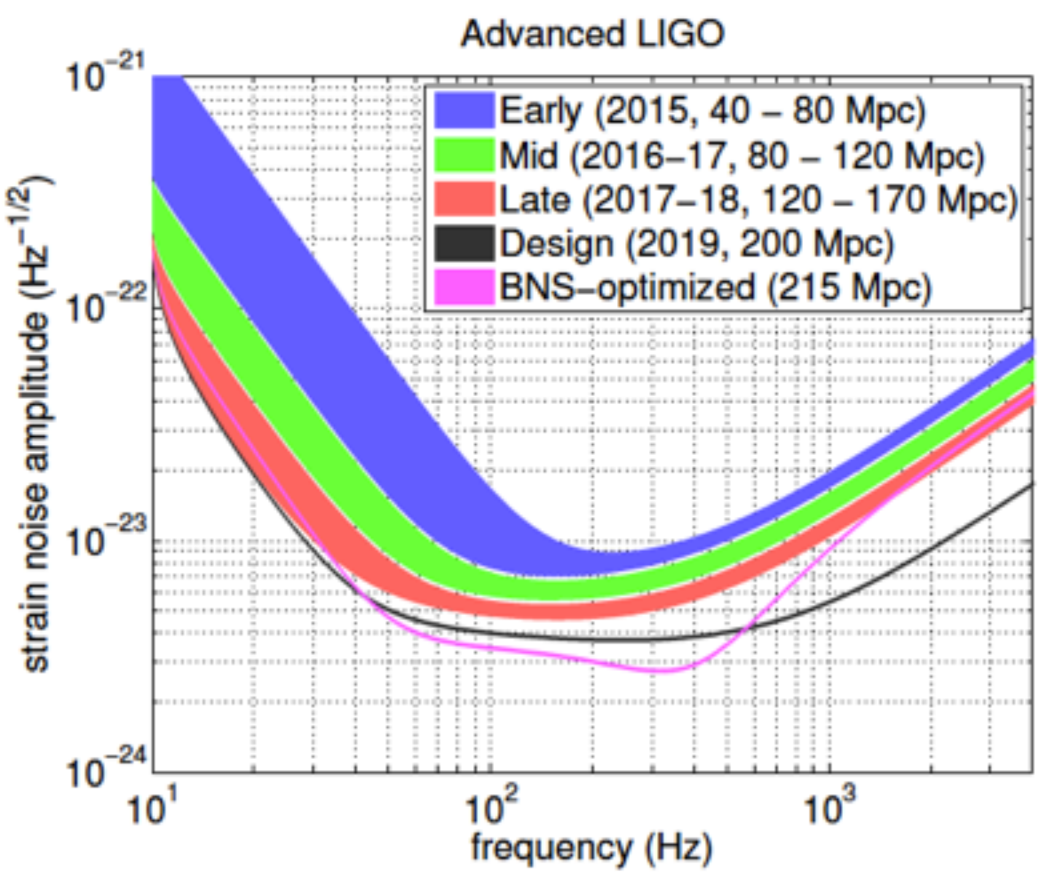
Detectors are sensitive to the amplitude instead of intensity

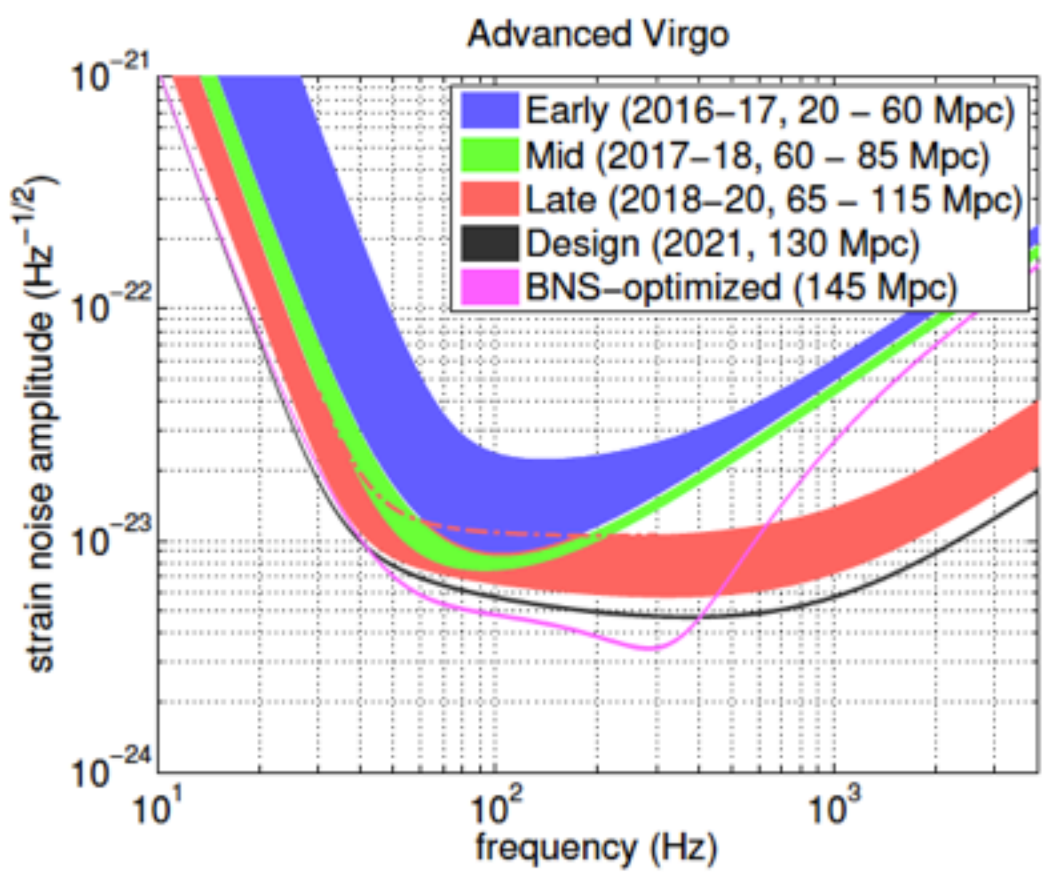
→ range is proportional to the sensitivity

→ number of events to the sensitivity³

Advanced Interferometer network

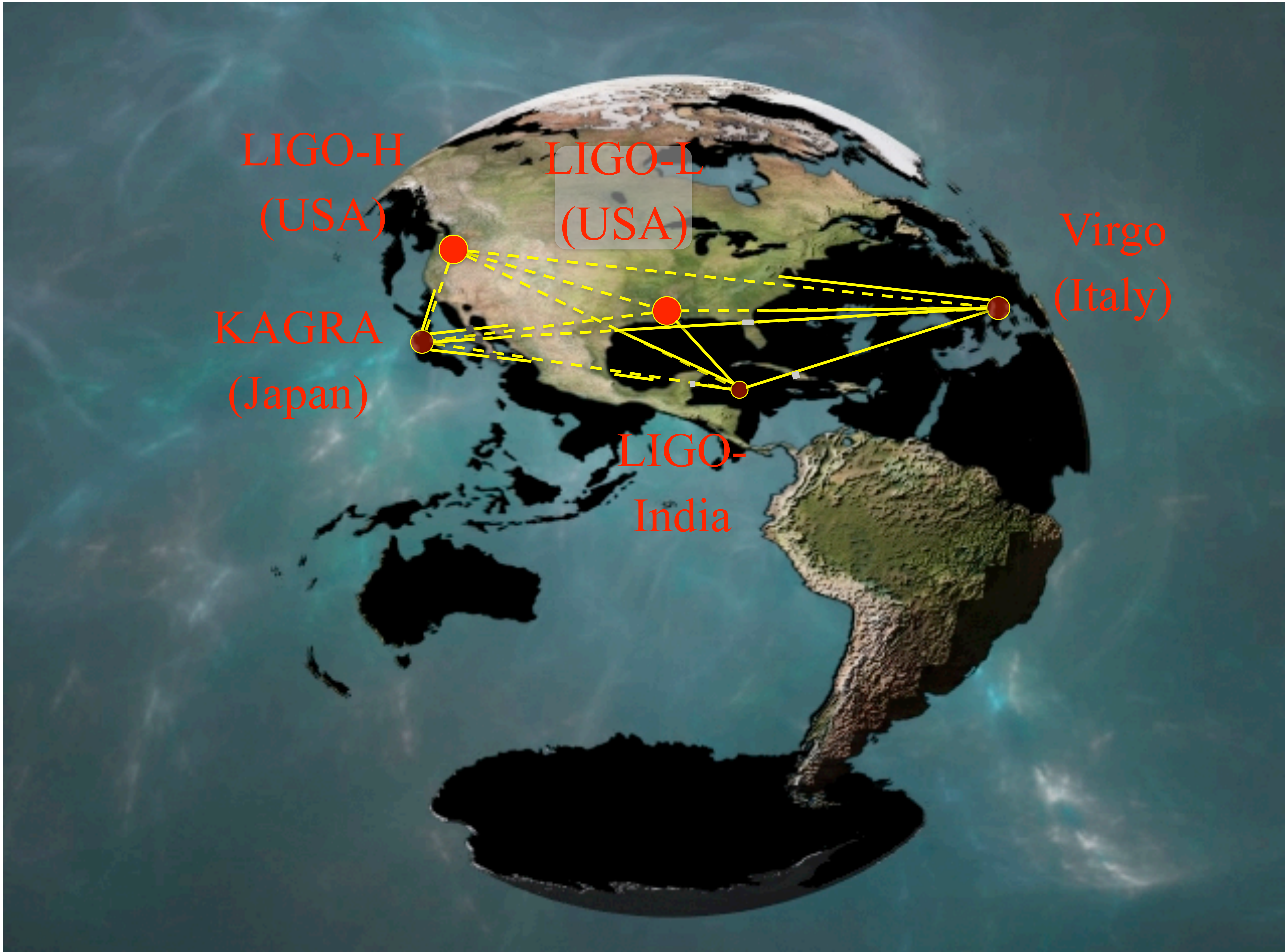
The advanced versions of the LIGO Virgo interferometers to enter in service in 2015





First direct detection expected before 2020 (2017-18 ?)

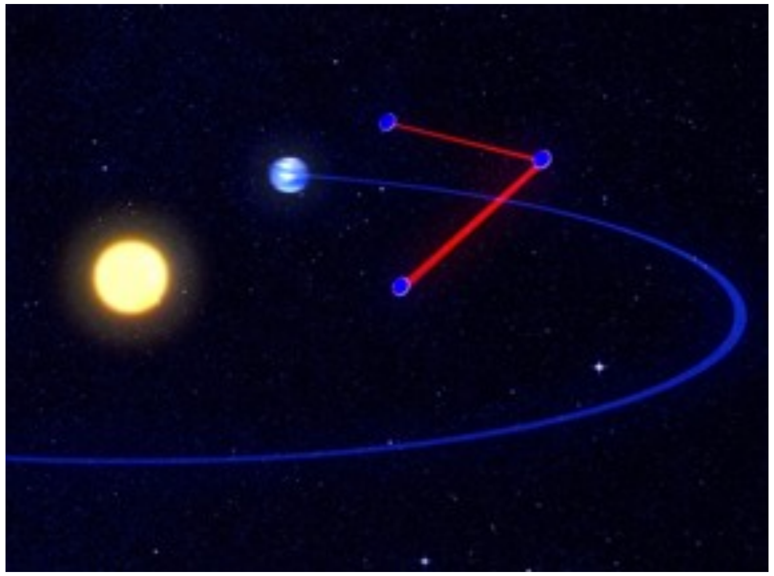
The Global Network c. 2020



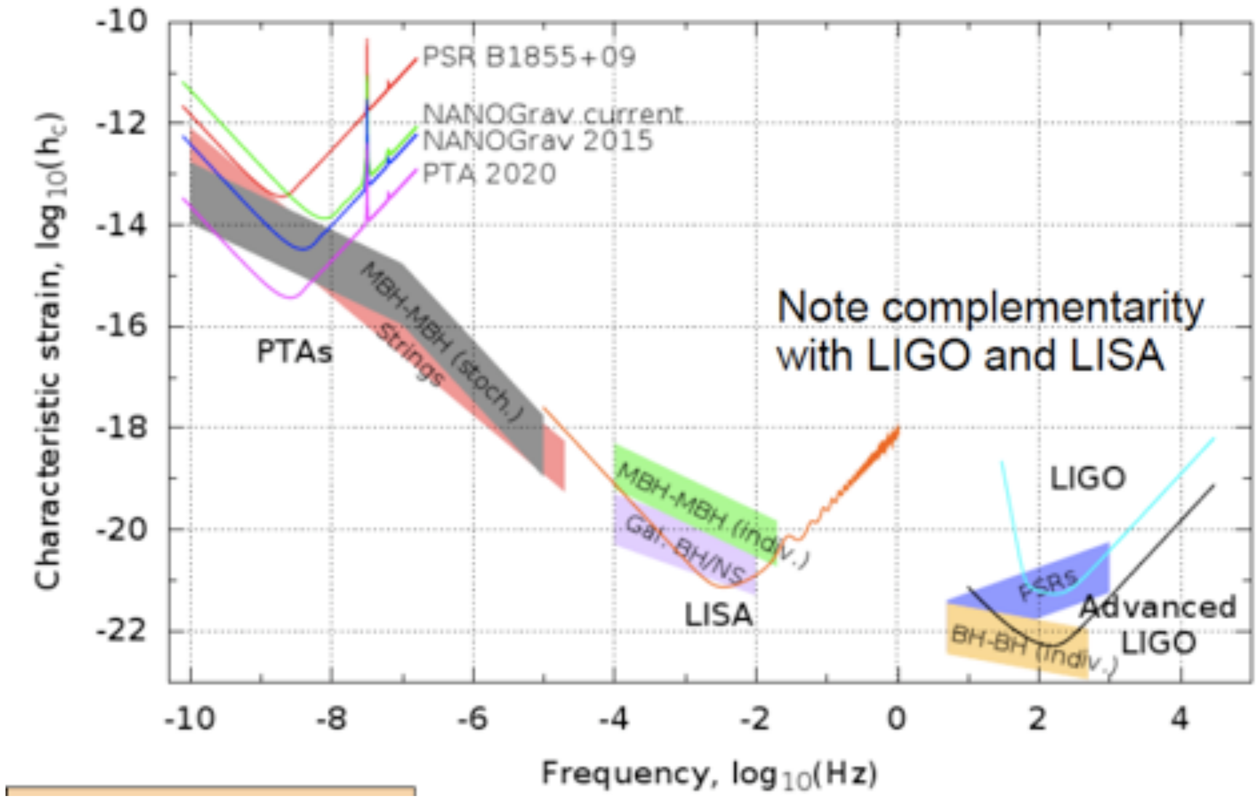
Other frequency bands

Pulsar Timing

eLISA



selected by ESA for L3
launch ~2034



(some) Motivation

Cosmological sources

- cosmic strings
- phase transitions
- inflation - preheating
- ?

Astrophysical sources

- binary systems of black holes and/or neutron stars
- supernovae
- highly spinning neutron stars

Gravitational Waves

Fundamental physics

- Window on very high energy physics
- Test General Relativity in the strong field regime
- Measure cosmo. parameters (dark energy eos?)
- Neutron star eos. Matter at high densities

Astrophysics

- Mass distrib of compact objects. Mass gap?
- Intermediate Mass Black Holes? Hierarchical mergers of galaxies.
- How do stellar mass compact binaries form and evolve?
- Origin of short gamma ray bursts?

Compact Binary Coalescences

Neutron stars

$M \lesssim 3M_{\odot}$

IMBH =?

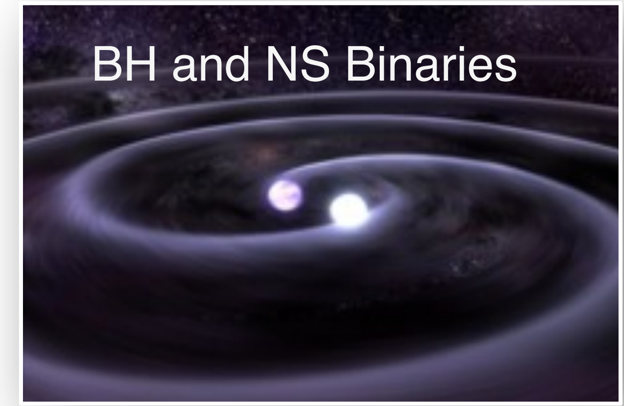
Mass

Stellar Mass Black Holes

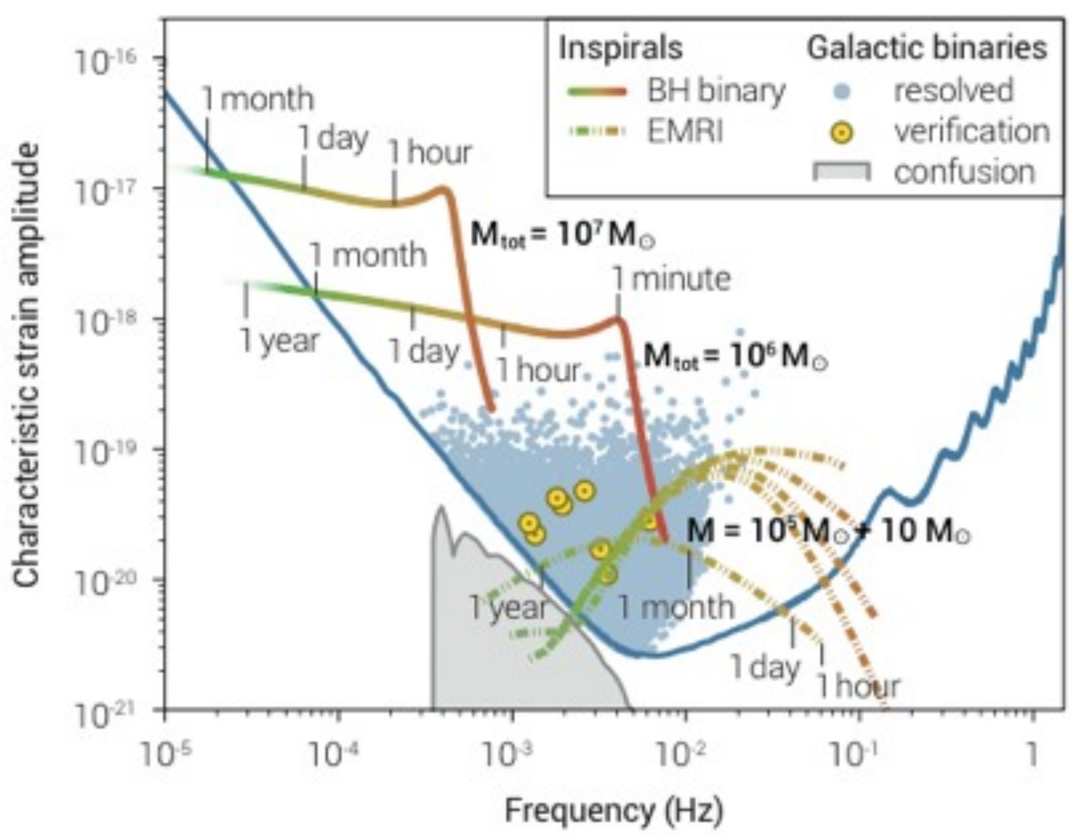
$5M_{\odot} \lesssim M \lesssim 10^2 M_{\odot}$

Super Massive Black Holes

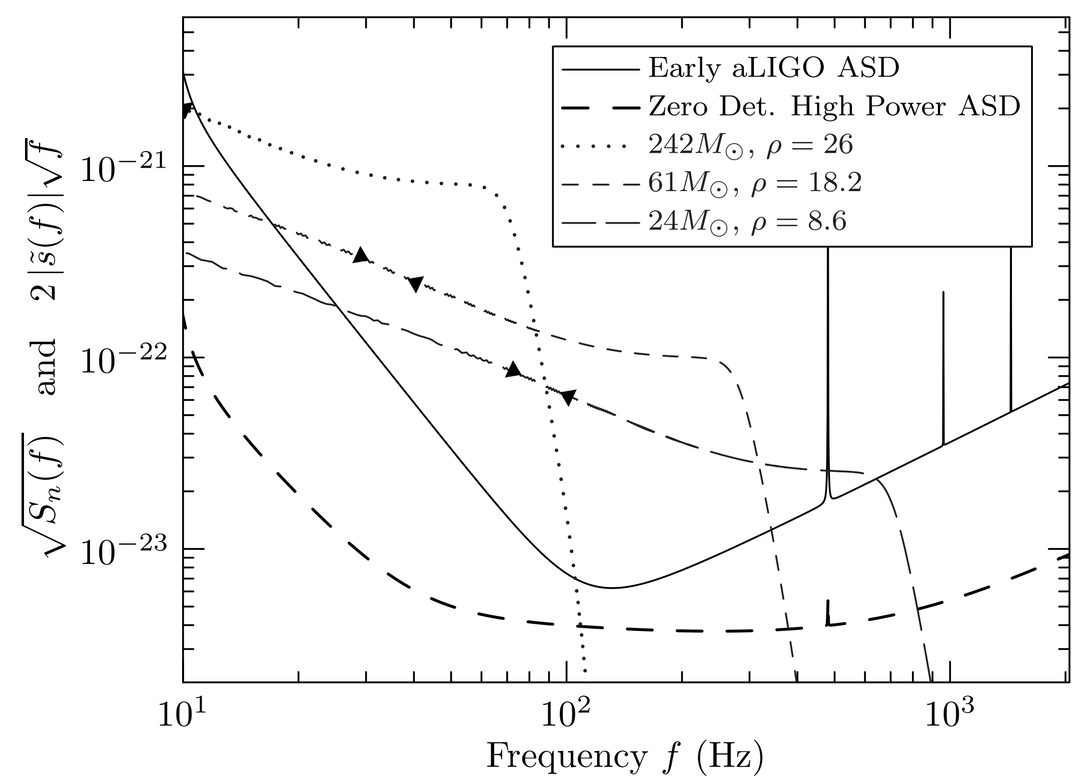
$10^6 M_{\odot} \lesssim M \lesssim 10^9 M_{\odot}$



eLISA: Super Massive Black Holes



Adv LIGO/Virgo
Stellar Mass Black Holes - Neutron Stars



Estimated rates

Epoch	Estimated Run Duration	$E_{GW} = 10^{-2} M_{\odot} c^2$ Burst Range (Mpc) LIGO	Virgo	BNS Range (Mpc) LIGO	Virgo	Number of BNS Detections	% BNS Localized within 5 deg^2	20 deg^2
2015	3 months	40 – 60	–	40 – 80	–	0.0004 – 3	–	–
2016–17	6 months	60 – 75	20 – 40	80 – 120	20 – 60	0.006 – 20	2	5 – 12
2017–18	9 months	75 – 90	40 – 50	120 – 170	60 – 85	0.04 – 100	1 – 2	10 – 12
2019+	(per year)	105	40 – 80	200	65 – 130	0.2 – 200	3 – 8	8 – 28
2022+ (India)	(per year)	105	80	200	130	0.4 – 400	17	48

Table 5. Detection rates for compact binary coalescence sources.

IFO	Source[a]	$\dot{N}_{low}$ yr^{-1}	$\dot{N}_{re}$ yr^{-1}	$\dot{N}_{high}$ yr^{-1}	$\dot{N}_{max}$ yr^{-1}
Initial	NS–NS	2×10^{-4}	0.02	0.2	0.6
	NS–BH	7×10^{-5}	0.004	0.1	
	BH–BH	2×10^{-4}	0.007	0.5	
	IMRI into IMBH			<0.001[b]	0.01[c]
	IMBH-IMBH			10^{-4}[d]	10^{-3}[e]
Advanced	NS–NS	0.4	40	400	1000
	NS–BH	0.2	10	300	
	BH–BH	0.4	20	1000	
	IMRI into IMBH			10[b]	300[c]
	IMBH-IMBH			0.1[d]	1[e]

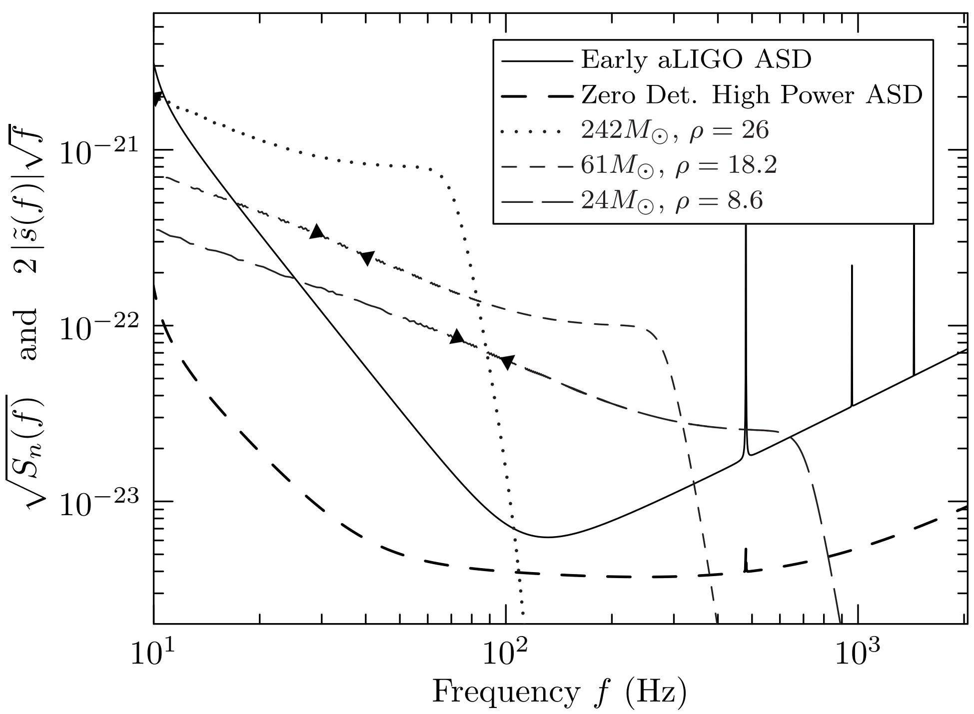
Class.Quant.Grav. 27 (2010)

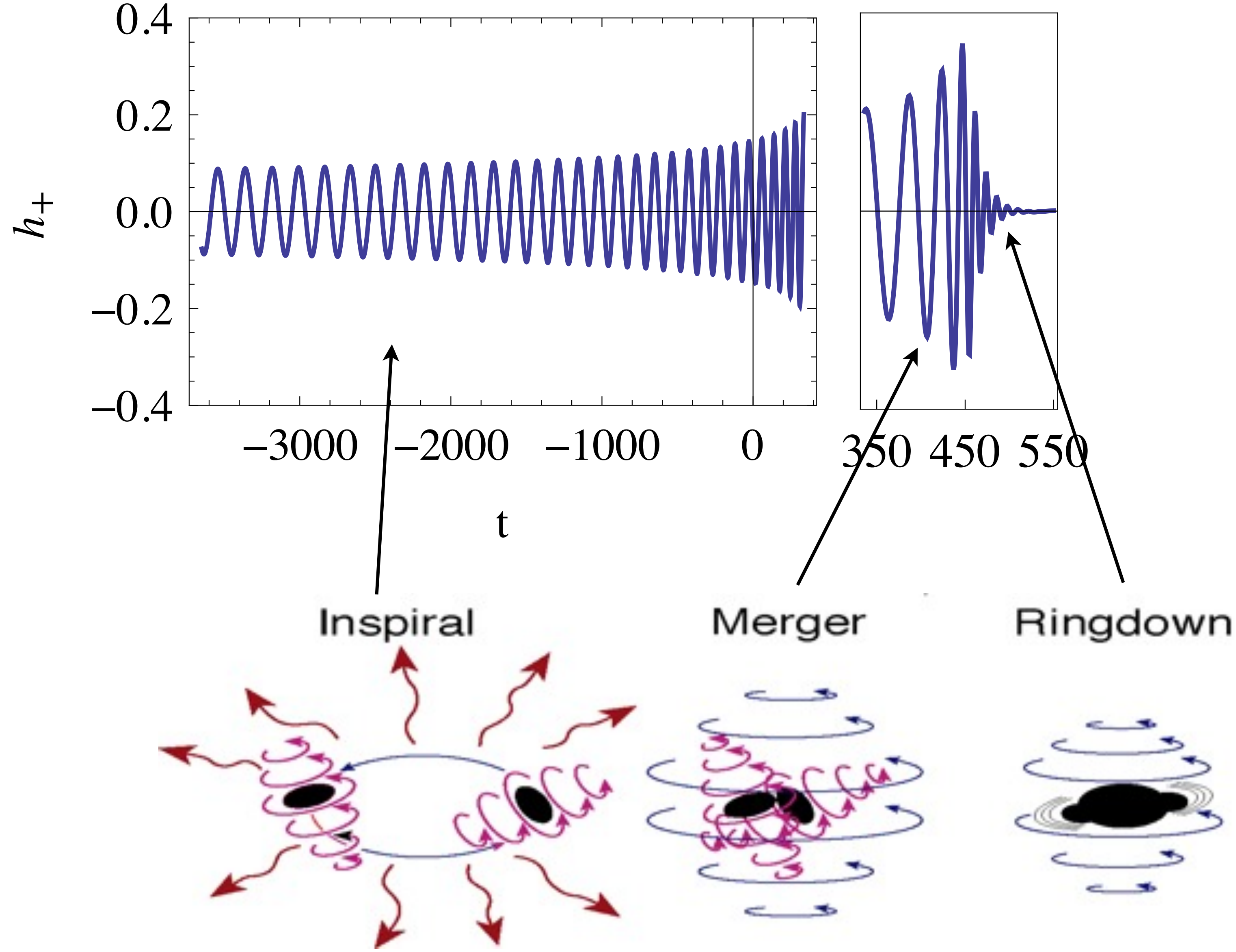
Progress

- Gravitational waves and Compact Binary Systems
- Tools for modeling Compact Binary Coalescences
- PhenomP: a closed form IMR model for precessing systems

Dynamics of Compact Binary Coalescences

Loses energy by GW emission
→separation decreases
(and frequency increases)





To extract the signal from the instrumental noise (matched filtering),
the waveform needs to be modeled with great accuracy

Intrinsic parameter space

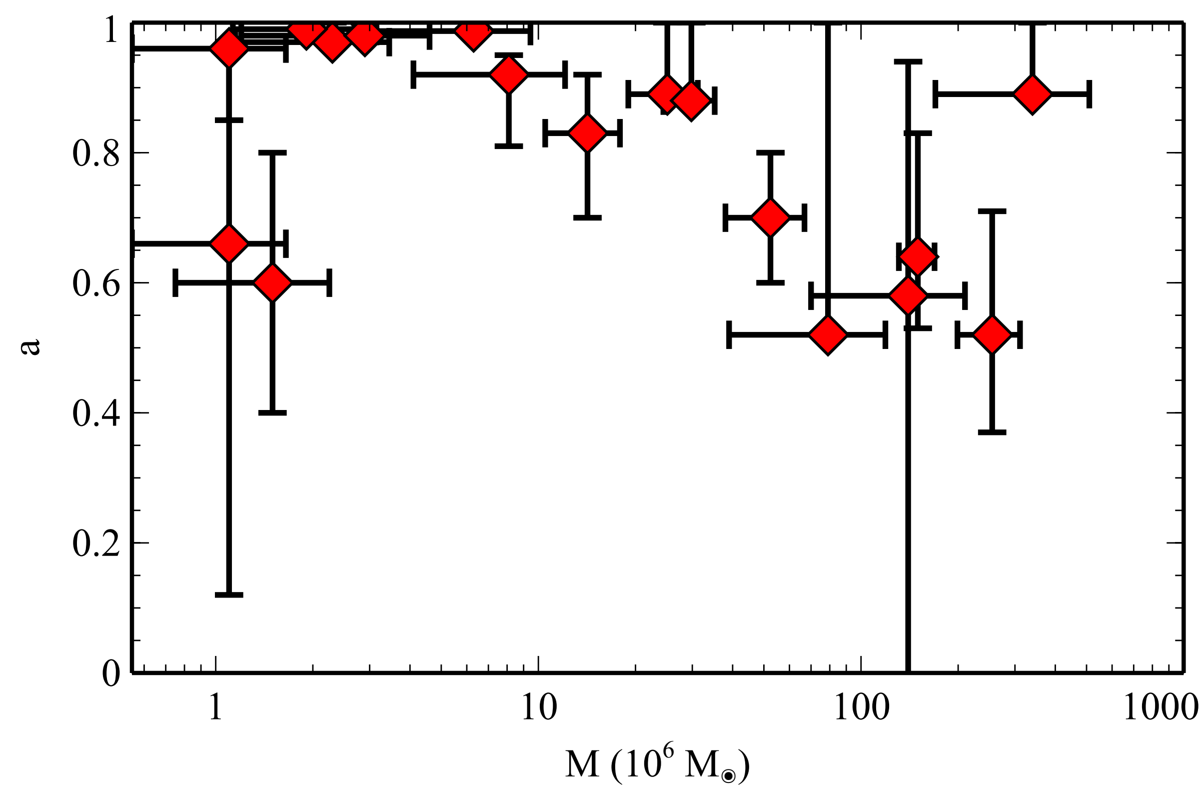
- Total mass M: scale factor
- Mass ratio q= m1/m2
- Spin ! 6 components in general

Neutron stars

$$a < 0.4$$

in binary systems $a < 0.04$

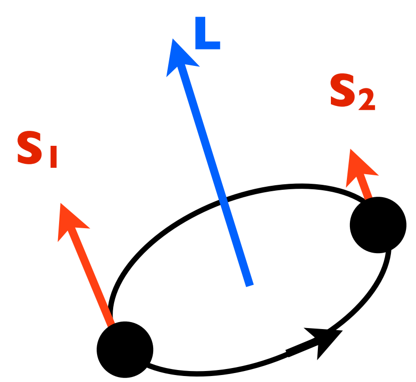
black holes generically have large spins



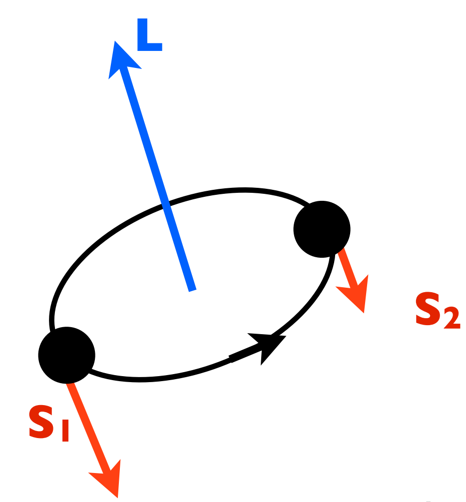
Reynolds 1302.3260 (2013)

Effect of Spin

The components of the spins that are orthogonal to the orbital plane change the inspiral rate



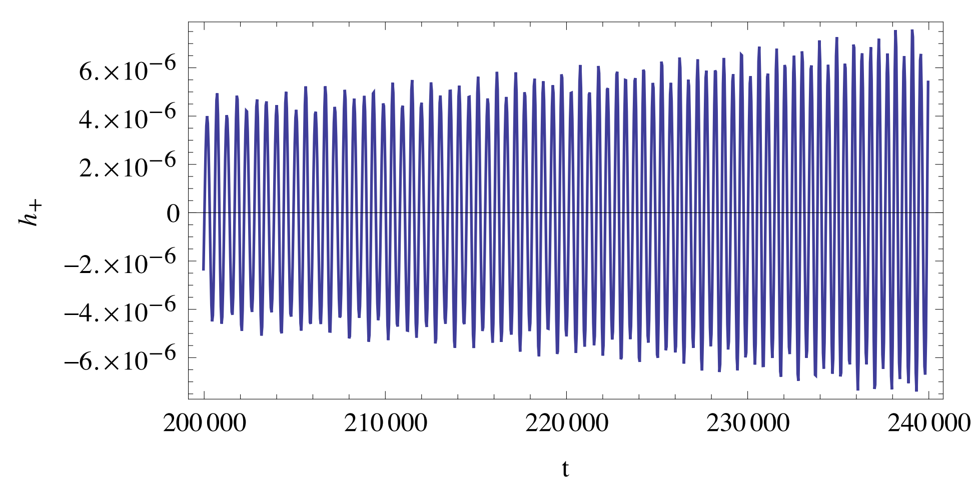
aligned spins
slower inspiral
more energy radiated

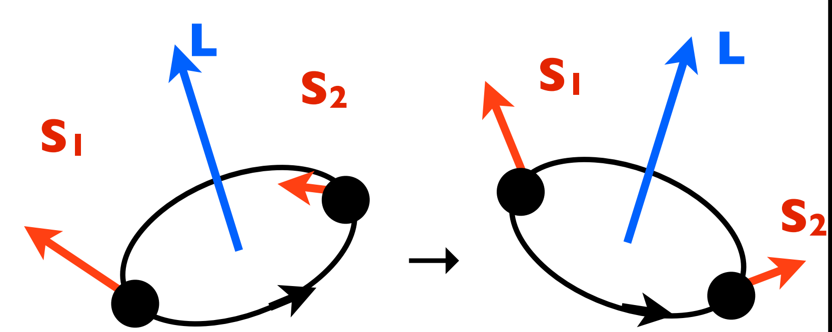


anti-aligned spins
slower inspiral
less energy radiated

Generic case: precession of the orbital plane

Modulations





CBC: modeling the inspiral

During the «slow» inspiral, while the objects are far from each other, a perturbative treatment is valid:

Newtonian estimate

post-Newtonian expansion in v/c

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\frac{Gm\mu}{r} \quad \text{i.e.} \quad \frac{v^2}{c^2} = \frac{R_s}{2r} \qquad R_s = 2\frac{Gm}{c^2}$$

- Purely analytical approach: iterate Einstein equations in harmonic coordinates

rewrite Einstein eqs

$$h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu} \longrightarrow$$

$$\partial_\mu h^{\alpha\mu} = 0 \quad \text{harmonic gauge}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}$$

$$\tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}$$

$\tau^{\mu\nu}$ stress-energy pseudo tensor of matter + gravitational fields

(also 2 different approaches ADM and EFT)

- The formalism is based on an elegant combination of post-Minkovskian, post-Newtonian et multipolar expansions

(see Living Review by Blanchet)

- To make the calculation tractable: effective description in terms of (spinning) point particles (regularisation UV)

- For parameter estimation, we need at least 3.5PN precision in the phase (corrections up to $(v/c)^7$)

State of the art in PN

state of the art for the phase:

- non-spinning: 3.5 PN (a lot of effort currently on 4PN)
- spin-orbit: 4 PN Marsat, Bohe, Blanchet, Buonanno (2013)
- spin-spin: only leading order 2PN

$$\frac{dE}{dt} = -\mathcal{F} \Longrightarrow \frac{d\omega}{dt} = \frac{-\mathcal{F}}{dE/d\omega}$$

$$x = \left(\frac{Gm\omega}{c^3}\right)^{2/3} = \mathcal{O}((v/c)^2)$$

$$E = -\frac{\mu c^2 x}{2}\left[1 + e_1\, x + e_{1.5}\, x^{3/2} + e_2\, x^2 + e_{2.5}\, x^{5/2} + e_3\, x^3 + e_{3.5}\, x^{7/2} + \mathcal{O}(1/c^8)\right]$$

$$\mathcal{F} = \frac{32 c^5}{5G} x^5 \nu^2 \left[1 + f_1\, x + f_{1.5}\, x^{3/2} + f_2\, x^2 + f_{2.5}\, x^{5/2} + f_3\, x^3 + f_{3.5}\, x^{7/2} + f_4\, x^4 + \mathcal{O}_{\rm NS}(1/c^8)\right]$$

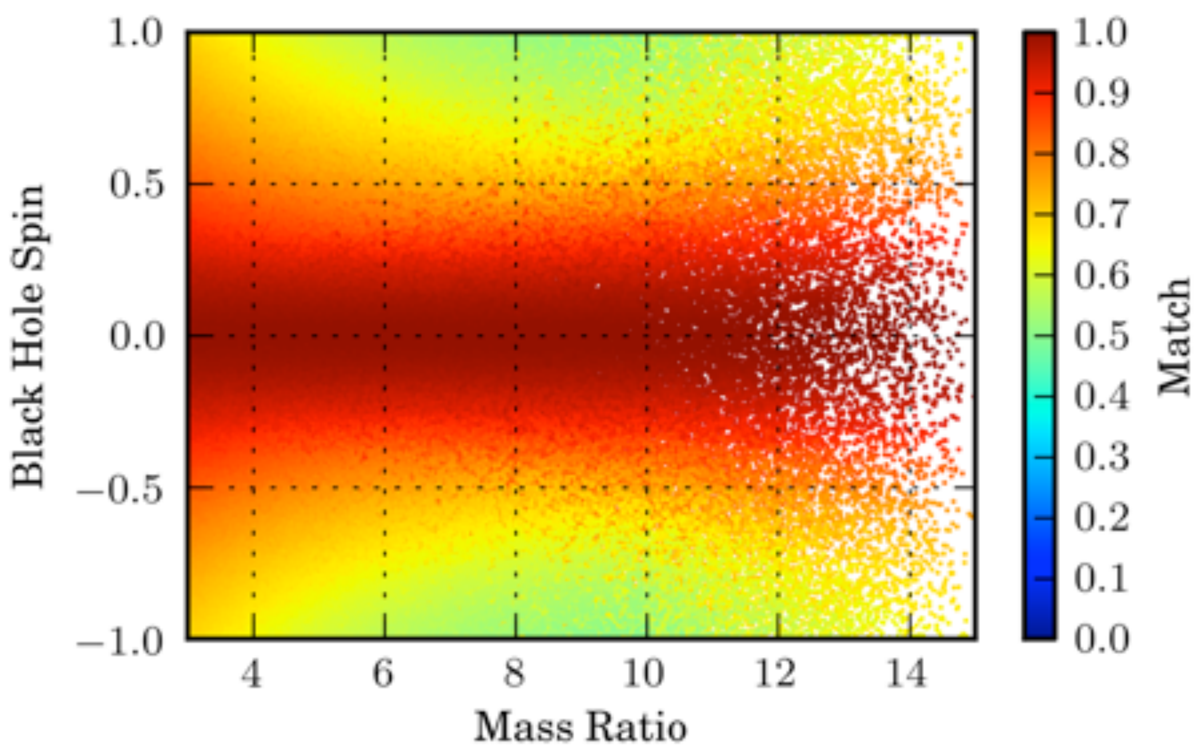
Assumption in the balance equation: constant spins!

Also Spin precession equations to Next-to-Next-to leading order

$$\frac{\mathrm{d}\mathbf{S}_1}{\mathrm{d}t} = \mathbf{\Omega}_1 \times \mathbf{S}_1$$

$$\mathbf{\Omega}_1 = \frac{1}{c^2}\mathbf{\Omega}_1^{\rm 1PN} + \frac{1}{c^4}\mathbf{\Omega}_1^{\rm 2PN} + \frac{1}{c^6}\mathbf{\Omega}_1^{\rm 3PN} + \mathcal{O}\left(\frac{1}{c^7}\right)$$

Bohe, Marsat, Blanchet, Faye (2013)



Nitz, Lundgren, Brown, Ochsner, Keppel, Harry (July 2013)

LIGO Technical Document

Studying the Merger

«Visible» for massive systems (say M≳12 $M_\odot$)

Non linearities become too strong: PN expansion breaks down
→ need to resort to Numerical Relativity
simulation of the full Einstein equations

Very expensive: O(1000) configs. only (a few 10^5 CPU hours/config)
Simulation of O(10) orbits. Going to low frequencies is very expensive.

$$\tau_{\rm coalescence} \approx \nu^{-1} f_{initial}^{-8/3}$$

Two main approaches:
- finite differences
- spectral codes

Intrinsic parameter space is 7D: mass ratio + 6 spin components. Impossible to sample

For DA purposes, we need analytical models calibrated to simulations

Modeling the ringdown

After the merger, we are left with a single perturbed BH decaying into Kerr. The system is well described by BH perturbation theory.

Evolution equation for perturbations of Kerr written in terms of $\psi = \psi_4(r, r, \theta, \phi)\rho^{-4}$

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2}\right]\frac{\partial^2\psi}{\partial\phi^2} - \Delta^2\frac{\partial}{\partial r}\left(\Delta^{-1}\frac{\partial\psi}{\partial r}\right)$$
$$- \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + 4\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\phi}$$
$$+ 4\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (4\cot^2\theta^2 + 2)\psi = 0$$

Teukolsky equation

Separate variables $\psi(t, r, \theta, \phi) = e^{-i\omega t}e^{im\phi}S(\theta)R(r)$ ω complex

(Radial equation+) angular equation $u = \cos\theta$

$$\frac{d}{du}\left[(1-u^2)\frac{dS}{du}\right] + \left[a^2\omega^2u^2 + 4a\omega u - 2 + A - \frac{(m-2u)^2}{1-u^2}\right]S = 0$$

Boundary conditions for the radial solution selects a discrete set of complex frequencies

Quasi Normal Modes $\omega_{l,m,n}$

$$\psi_4(t, \theta, \phi) = \sum_{l,m,n} \tilde{\psi}_{lmn} e^{-i\omega_{lmn}t} S_{lmn}(\theta) e^{im\phi}$$

Inspiral-Merger-Ringdown models

For data analysis purposes, we need models that cover the full coalescence and that are fast to evaluate (purely analytical or solving ODEs)

Two main strategies have been proposed and implemented so far

- Effective One Body formalism (Damour, Buonanno (98))
  resummation of the PN results
  map the two body problem to the motion of a test particle in a deformed Kerr metric
  factorized waveform
  calibration to NR
  connection to ringdown: sum of quasinormal modes

- Phenomenological models
  PN at low frequencies
  ansatz fitted to NR simulations for the merger
  effective spin parameter
  connection to ringdown

⟶ Phenom B/C models for aligned spins

Ajith+ CQG 2007, Ajith+ PRD 2008
Ajith+ PRL 2011, Santamaria+ PRD 2010

Progress

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IMR Phenom models: aligned spin

$$\tilde{h}_{\rm phen}(f) = A_{\rm phen}(f)\, e^{i\Phi_{\rm phen}(f)}$$

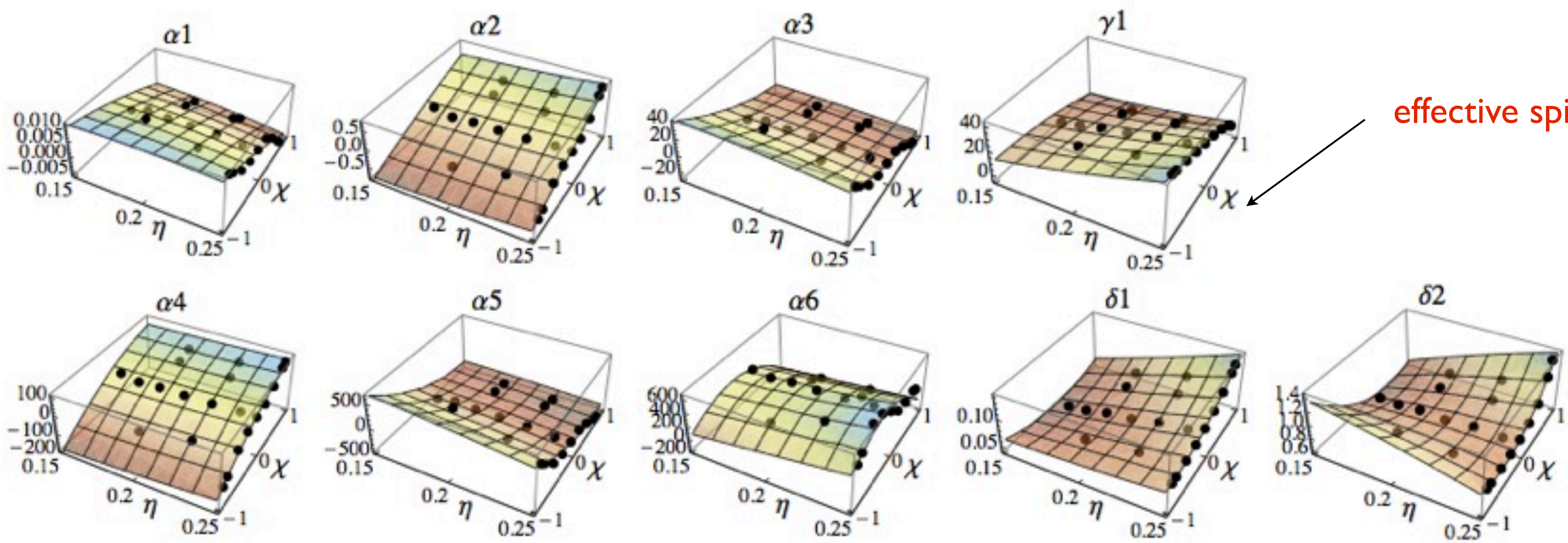
$$\Phi_{\rm phen}(f) = \psi^{22}_{\rm SPA} w^-_{f_1} + \psi^{22}_{\rm PM} w^+_{f_1} w^-_{f_2} + \psi^{22}_{\rm RD} w^+_{f_2}$$

PN RD

$$\psi^{22}_{\rm PM}(f) = \frac{1}{\eta}\Big(\alpha_1 f^{-5/3} + \alpha_2 f^{-1} + \alpha_3 f^{-1/3} + \alpha_4 + \alpha_5 f^{2/3} + \alpha_6 f\Big)$$

$$w^{\pm}_{f_0} = \frac{1}{2}\left[1 \pm \tanh\left(\frac{4(f-f_0)}{d}\right)\right]$$

Fit of the dependence of the phenomenlological parameters on the physical parameters via hybrid waveforms



Ajith+ CQG 2007, Ajith+ PRD 2008
Ajith+ PRL 2011, Santamaria+ PRD 2010

aligned IMR Phenom: effective spin

In principle 3 intrinsic parameters: $\eta = \frac{m_1 m_2}{(m_1+m_2)^2}, \chi_1 = \frac{S_1}{m_1^2}, \chi_2 = \frac{S_2}{m_2^2}$

Idea: capture the main features of aligned spin waveforms with as little new parameters as possible (the more params there are, the more expensive the DA). On the other hand, prevents from measuring individual spins...

Fourier domain PN phase:

$$\Psi(f) = \frac{3}{128\eta v^5}\left\{1 + v^2\left[\frac{3715}{756} + \frac{55\eta}{9}\right] - v^3\left[16\pi - \left(\frac{113}{3} - \frac{76\eta}{3}\right)\chi_s - \frac{113\delta}{3}\chi_a\right]\right\} + \dots$$

leading order effect of spin

$$\chi_s = (\chi_1 + \chi_2)/2$$
$$\chi_a = (\chi_1 - \chi_2)/2$$

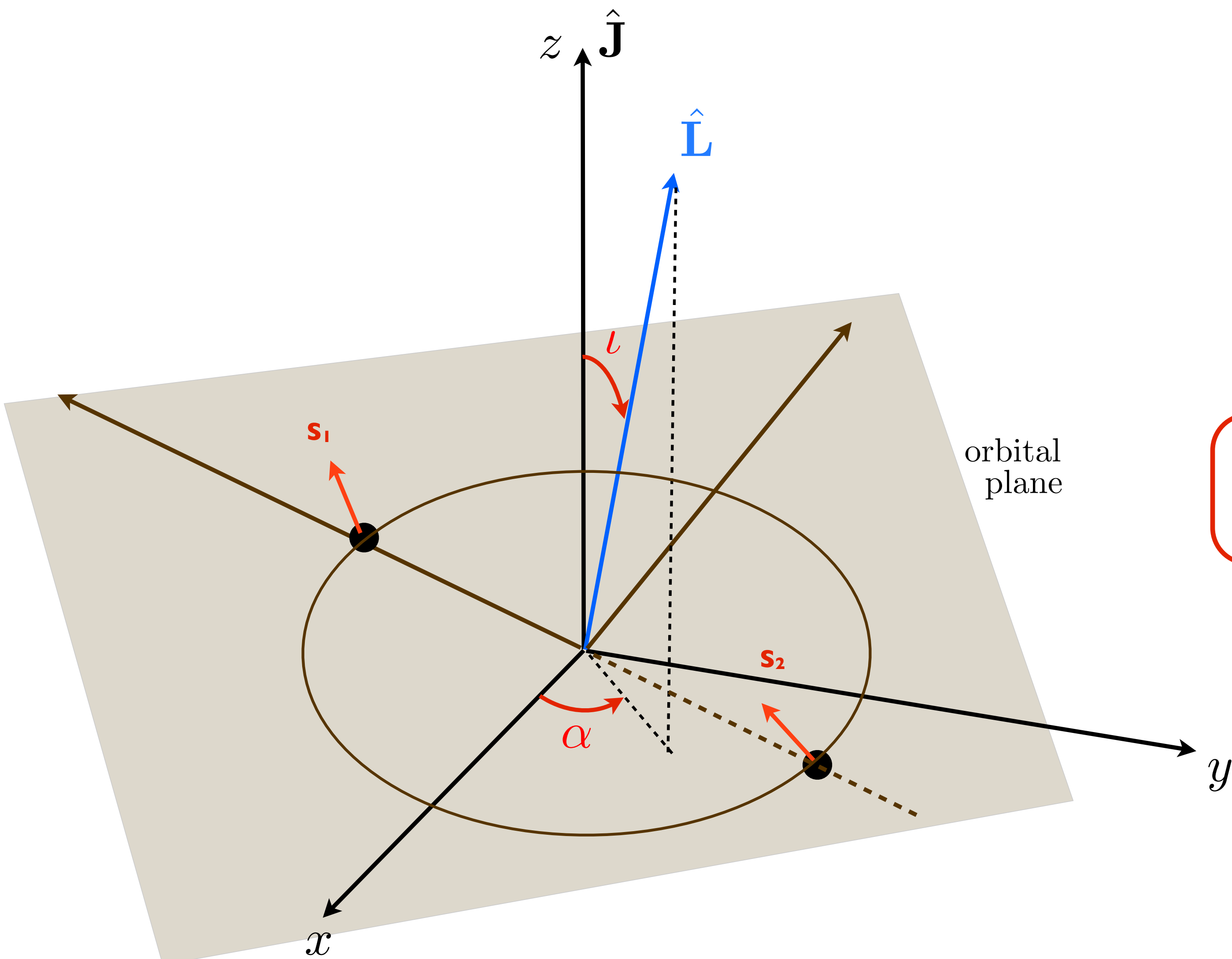
The effective parameter $\chi \equiv \chi_s + \delta\chi_a - \frac{76\eta}{113}\chi_s$

is sufficient to reproduce the leading order effect of spin in the phase. One can rewrite the higher orders in terms of it plus a correction that is ignored.

In fact, for historical reasons, slightly different choice...

(cf Pürrer et al (2013))

Dynamics of precession



$$\mathbf{J} = \mathbf{L} + \mathbf{S_1} + \mathbf{S_2}$$

L orbital angular momentum

3 timescales: $t_{\rm orb} \ll t_{\rm prec} \ll t_{\rm rad.reac.}$

On the orbital timescale: **J**, **L**, $\mathbf{S_1}$, $\mathbf{S_2}$ fixed

On the precessional timescale: **L**, $\mathbf{S_1}$, $\mathbf{S_2}$ precess around **J** which remains fixed

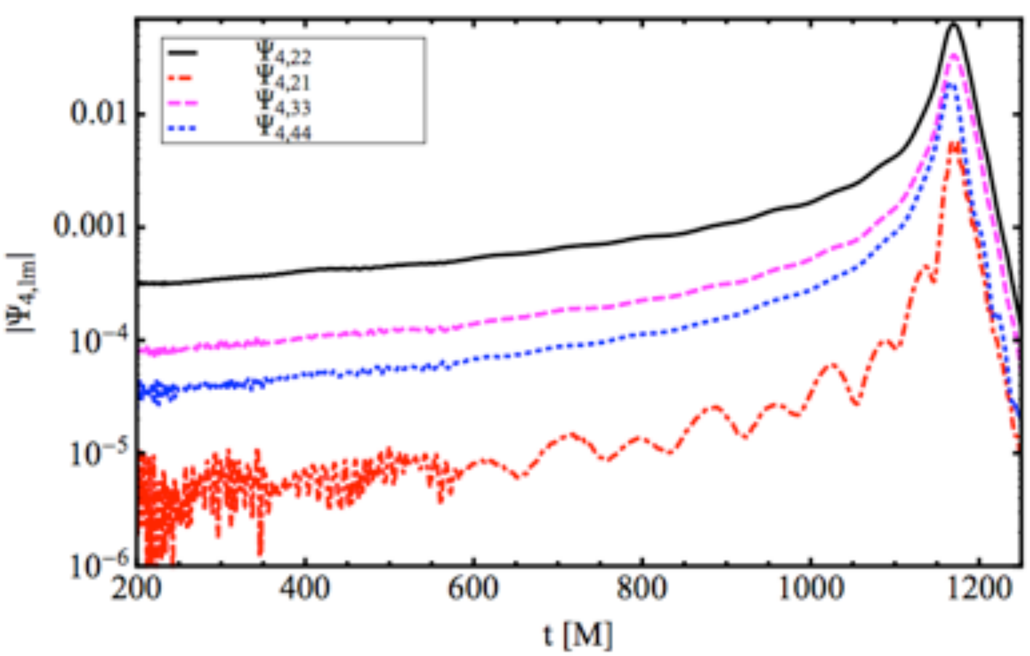
$$\frac{\mathrm{d}\mathbf{S}_1}{\mathrm{d}t} = \mathbf{\Omega}_1 \times \mathbf{S}_1 \qquad \Omega_1 = \frac{1}{c^2}\Omega_1^{\rm 1PN} + \frac{1}{c^4}\Omega_1^{\rm 2PN} + \frac{1}{c^6}\Omega_1^{\rm 3PN} + \mathcal{O}\left(\frac{1}{c^7}\right) \longrightarrow \dot{\alpha}(t)$$

On the radiation reaction timescale: **J** and **L** shrink but in most cases the orientation of **J** remains constant. ι varies

Factorizing precession effects

Idea: one can factorize the effect of precession by going to a non inertial frame in co-rotation with the system. «Quadrupole alignment»

Precessing waveform + appropriate rotation R(t) ≃ Non Precessing waveform



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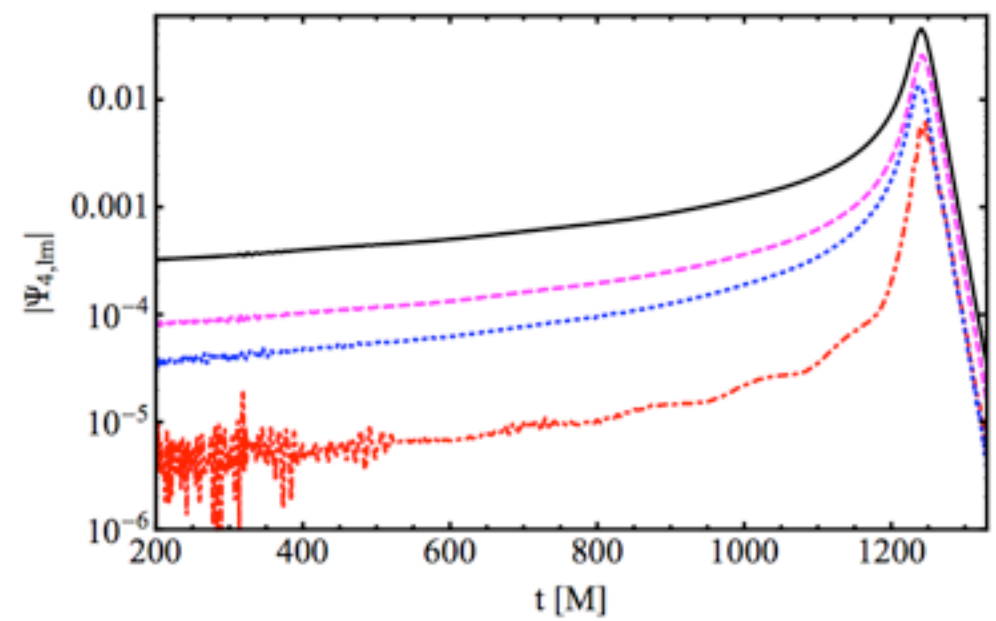


Schmidt et al. (2011, 2013), O'Shaughnessy et al. (2011, 2013), Boyle et al. (2011, 2013), Pekowsky et al. (2013)

The appropriate rotation can be read off the precessing waveform by following the direction that instantaneously maximizes the radiated power.

This closely follows the orbital angular momentum **L**.

⟶ One can model a priori the rotation by solving the precessional dynamics (ι, α)

Twisting up non precessing waveforms

One cheap(er) way of modeling precessing wfs is to model the evolution of **L** i.e. of (ι, α)

- deduce R(t) from EOB dynamics → EOB
- analytical PN prescription → PhenomP

and then twist up a non precessing waveform

$$\dot{\epsilon} = \dot{\alpha}\cos\iota$$

$$h^P_{2m}(t) = e^{-im\alpha} \sum_{|m'|=2} e^{im'\epsilon} d^2_{m',m}(-\iota) h_{2,m'}(t),$$

precessing modes

angle dependent factors

non precessing waveform modes (PhenomC)

- PN angles with NNLO spin-orbit corrections, continued through merger
- model formulated in the frequency domain (faster DA) using the SPA (even through merger!)
- Uses approximate degeneracies 6 → 2 spin params
- Note that no NR precessing simulation was used to formulate the model

Hannam et al. submitted to **PRL**.
PhenomP coded up in LAL

Effective spin parameters

Here again, the idea is to minimize the number of «extra» parameters with respect to non precessing models, i.e. to capture the main features of precessing wfs with as little new parameters as possible.

The quantity that affects the phase the most is the precessional speed $\dot{\alpha}$. Its leading order in PN can again be described by some combination of the spins, but it is not constant!

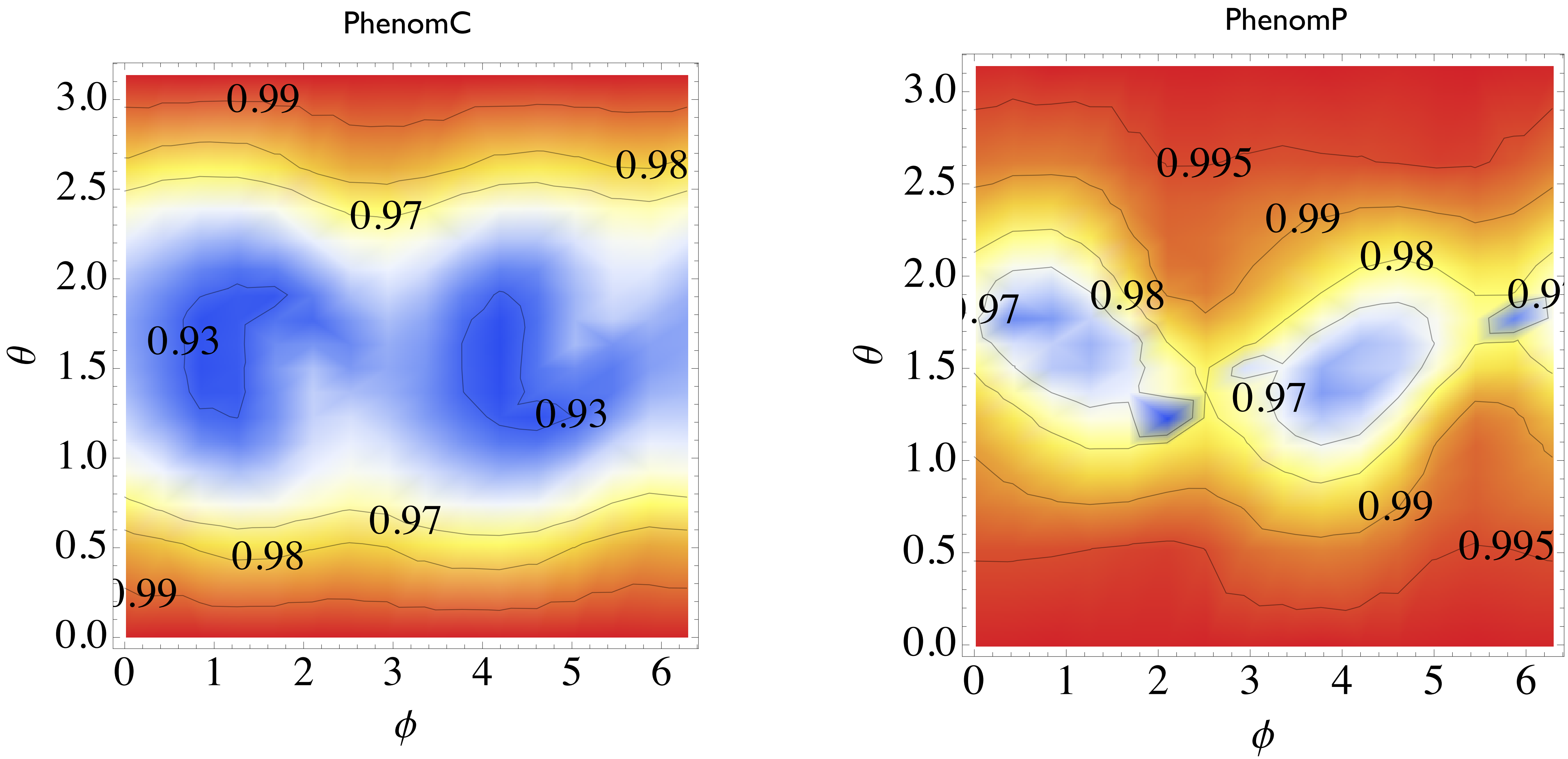
We use the following strategy to restrict ourselves to ONE extra spin parameter:

- consider a single spin system
- average the PN precessional equations over the orientation of the spin in the orbital plane
- the averaged equations now only depend on χ_p and the effective aligned spin $\chi_{\rm eff}$

Our new parameter has a simple interpretation in the single spin case. In the double spin case, we expect that some value will allow to capture the main effects. (presumably the one that reproduces the averaged LO of $\dot{\alpha}$)

Note that from the point of view of data analysis, this doesn't just mean one extra parameter: source orientation and polarization now have to be taken into account!

PhenomP: effectualness study



Fitting Factors against a PN-NR hybrid waveform with 50M, fixed polarization, q=3, single spin .75 in the plane

Other systems investigated:
single spin 1<q<3, double spins q=2
lower and higher masses

$$\langle h_1 | h_2 \rangle = 4\,\mathrm{Re} \int_{f_{\min}}^{f_{\max}} \frac{\tilde{h}_1(f)\tilde{h}_2^*(f)}{S_n(f)} df.$$

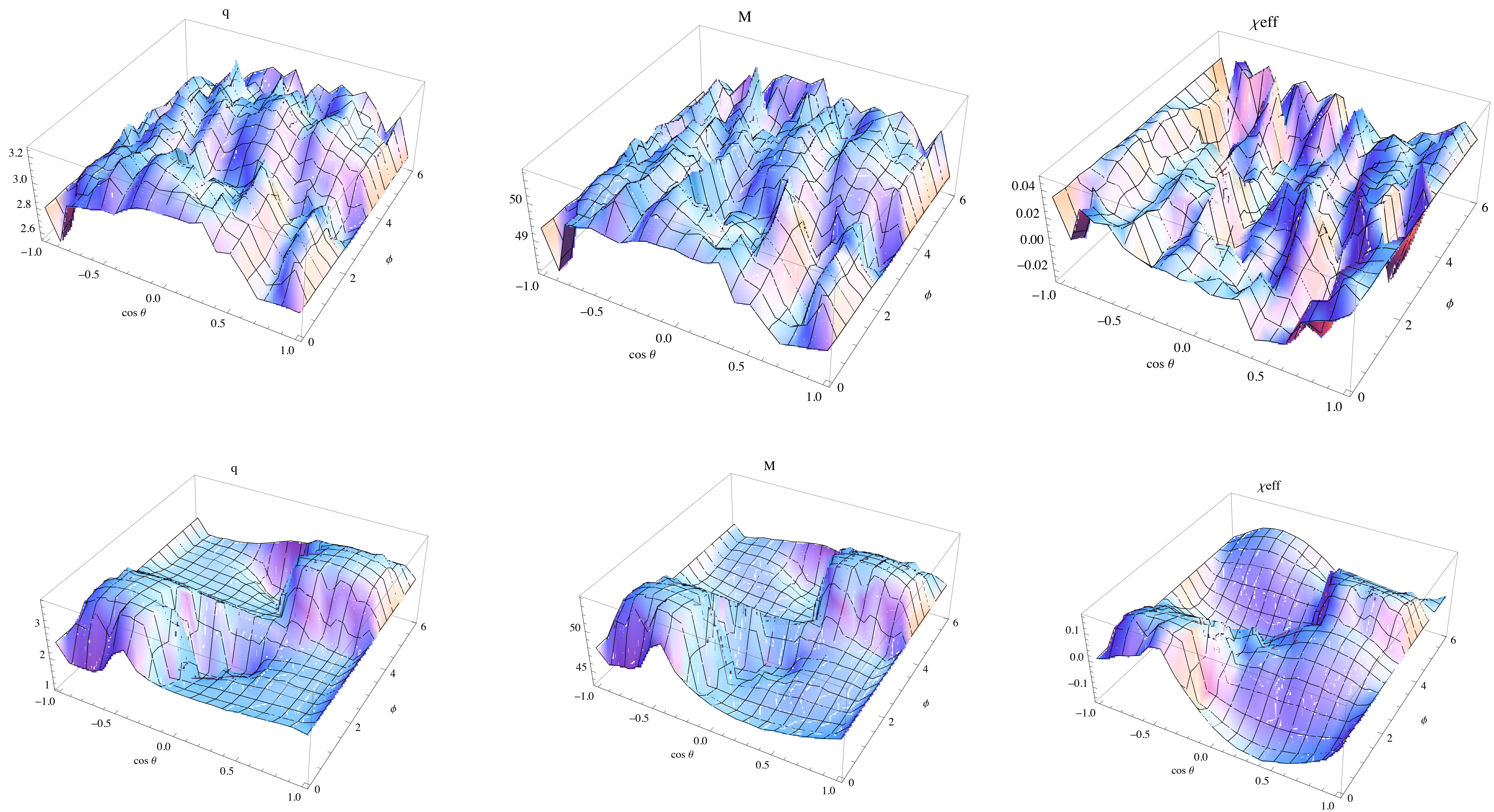
Worth trying a precessing search? (probably not)

Fitting factor = overlap optimized over the whole freedom in the model

PhenomP: parameter bias study

PhenomP

PhenomC

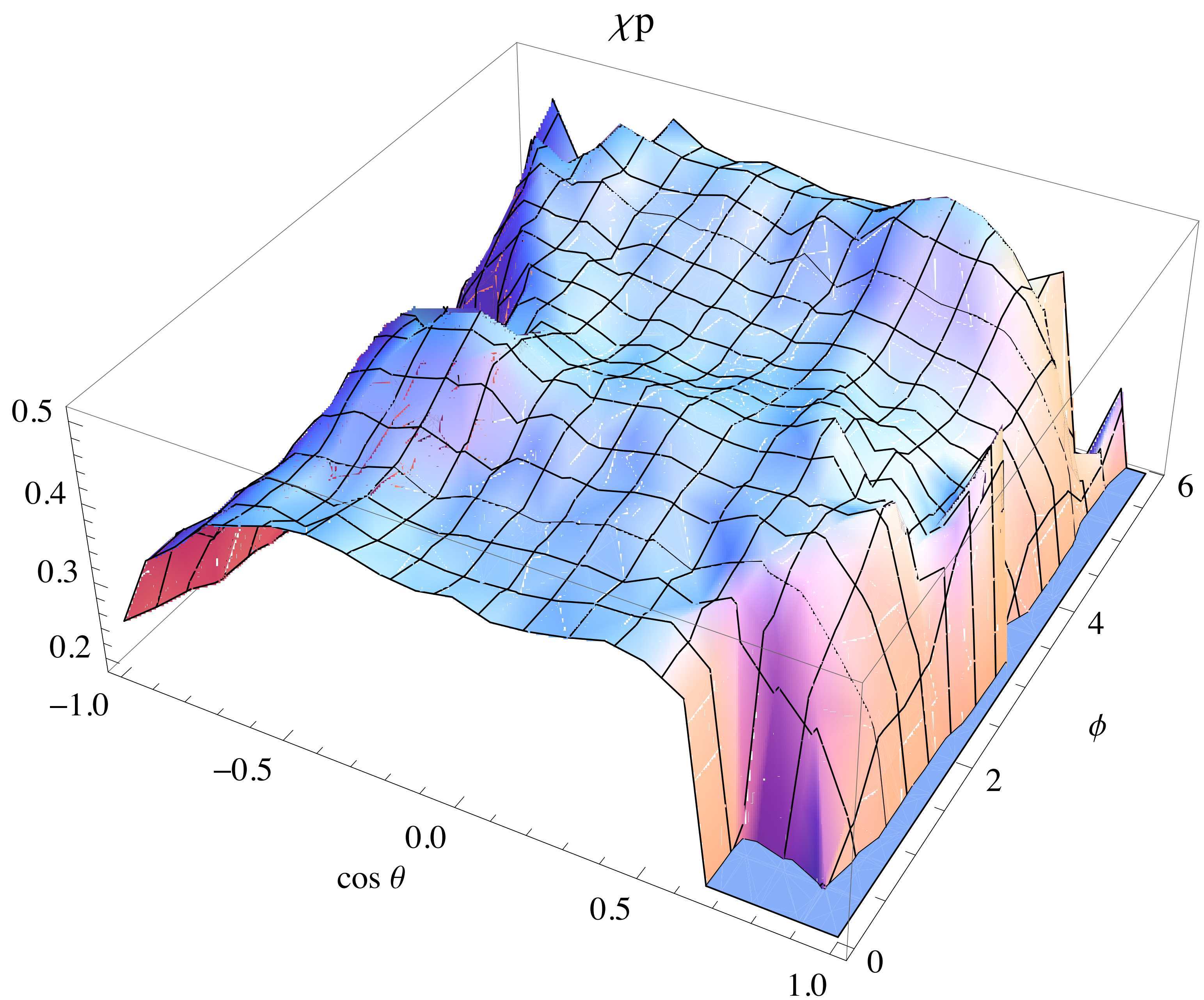


q=3

M=50

χ_{eff}=0

PhenomP: parameter bias study



Best fits obtained with effective precessing spin different from 0 (away from the poles)

Perspectives

The first direct detection of gravitational waves is expected within the next years with the LIGO-Virgo network of interferometers. Compact binary coalescences are likely to be the first source to be observed.

The waveforms need to be modeled with great precision to extract as much physics as possible. This is done via a combination of analytical and numerical methods.

One of the main current challenges lies in the modeling of precession through the whole coalescence. We have produced a first closed form model, PhenomP which looks promising.

Next steps

- calibrate the rotation during the merger
- add spin parameters?
- refine the underlying non precessing model (larger q)

Progress of the spin PN computations: dynamics

We redefine our spin variable as $S \equiv c\, S_{\rm phys} = \chi G m^2$

so that S is of Newtonian order for maximally spinning compact objects.

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = A^i_{\rm N} + \frac{1}{c^2} A^i_{\rm 1PN} + \frac{1}{c^3} {A_S^i}_{\rm 1.5PN} + \frac{1}{c^4}\left[A^i_{\rm 2PN} + {A_{SS}^i}_{\rm 2PN}\right] + \frac{1}{c^5}\left[A^i_{\rm 2.5PN} + {A_S^i}_{\rm 2.5PN}\right] + \frac{1}{c^6}\left[A^i_{\rm 3PN} + {A_{SS}^i}_{\rm 3PN}\right] + \frac{1}{c^7}\left[A^i_{\rm 3.5PN} + {A_S^i}_{\rm 3.5PN}\right] + \mathcal{O}(8)$$

LO Spin-Orbit ($1/c^3$):

Barker and O'Connell (75, 79)
Goldberger, Rothstein (06) (EFT approach)

NLO Spin-Orbit ($1/c^5$):

Tagoshi, Ohashi, Owen (98, 01)
Blanchet, Buonanno, Faye (06)
Damour, Jaranowski, Schäfer, (08) (ADM formalism)
Levi (10), Porto (10) (EFT)

Spin-Spin effects:

LO ($1/c^4$): Kidder, Will, Wiseman, (93)
Porto (05) (EFT)
Buonanno, Faye, Hinderer (13)
NLO ($1/c^6$): Steinhoff, Hergt, Schäfer (08,10) (ADM)
Porto, Rothstein (10), Levi (11) (EFT)
NNLO ($1/c^8$) spin1-spin2:
Hartung, Steinhoff (11) (ADM)
Levi (12) (EFT)

Here we compute the 3.5PN spin-orbit (linear in spin) correction together with the evolution equations for the spins

NNLO ($1/c^7$):

Hartung Steinhoff (11) (ADM)
Marsat, Bohe, Faye, Blanchet, (12)

Progress of the spin PN computations: Radiation

So far, a wave generation formalism has only been derived in the harmonic gauge formulation (although EFT on the way (cf Porto (06))

$$\mathcal{F} = \frac{32c^5}{5G} x^5 \nu^2 \left[1 + f_1\, x + f_{1.5}\, x^{3/2} + f_2\, x^2 + f_{2.5}\, x^{5/2} + f_3\, x^3 + f_{3.5}\, x^{7/2} + \mathcal{O}(4) \right]$$

For the flux

Spin-Orbit effects

LO ($1/c^3$): Kidder, Will, Wiseman (93, 95)

NLO ($1/c^5$): Blanchet, Buonanno, Faye (06)

NNLO ($1/c^7$): Bohe, Marsat, Blanchet, (13)

Tail SO effects

LO ($1/c^6$): Blanchet, Buonanno, Faye (06)

NLO ($1/c^8$): Marsat, Bohe, Blanchet, Buonanno

Spin-Spin effects

LO ($1/c^4$): Mikoczi, Vasuth, Gergely (05)

For the polarizations

SO LO ($1/c^3$): Kidder, Will, Wiseman (93, 95)
Arun, Buonanno, Faye, Ochsner (09)

SS LO ($1/c^4$): Kidder, Will, Wiseman (95, 96) Spin1-Spin2
Buonanno, Faye, Hinderer Spin1-Spin1

tail LO ($1/c^6$): Blanchet, Buonanno, Faye (06)

Analytical approach to the ringdown

After the merger, we are left with a single perturbed BH decaying into Kerr.
The system is well described by BH perturbation theory

Evolution equation for perturbations of Kerr written in terms of $\psi = \psi_4(r, r, \theta, \phi)\rho^{-4}$

$$\rho = -(r - ia\cos\theta)^{-1}$$

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2}\right]\frac{\partial^2\psi}{\partial\phi^2} - \Delta^2\frac{\partial}{\partial r}\left(\Delta^{-1}\frac{\partial\psi}{\partial r}\right)$$
$$- \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + 4\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\phi}$$
$$+ 4\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (4\cot^2\theta^2 + 2)\psi = 0$$

Teukolsky equation

$$\Sigma = r^2 + a^2\cos^2\theta$$
$$\Delta = r^2 - 2Mr + a^2$$

(Let us assume for now that we know the final spin)

Solving the Teukolsky equation

Separate variables $\psi(t,r,\theta,\phi) = e^{-i\omega t} e^{im\phi} S(\theta) R(r)$ ω complex

(Radial equation+) angular equation $u = \cos\theta$

$$\frac{d}{du}\left[(1-u^2)\frac{dS}{du}\right] + \left[a^2\omega^2 u^2 + 4a\omega u - 2 + A - \frac{(m-2u)^2}{1-u^2}\right] S = 0$$

We are looking for eigenfunctions of this operator which depends on $c = a\omega, \text{and } m$

These eigenfunctions are known as spheroidal harmonics. For a given c and m, discrete family of solutions $S_{lm}(c;u)$

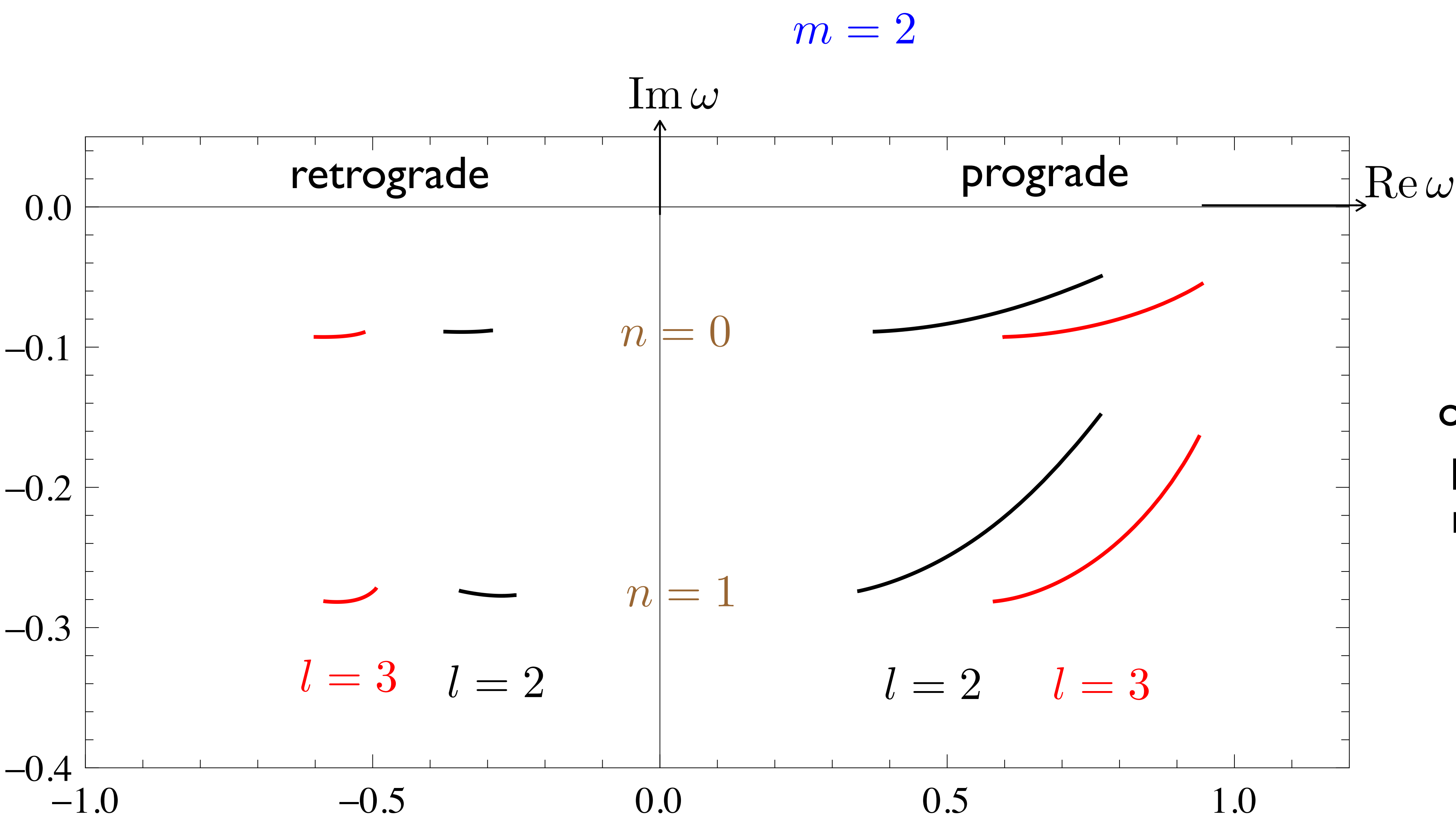
(cf Berti et al. 06)

Except at some discrete values of c, these form a complete non-orthogonal basis of the functions on the sphere

$$\int_{-1}^{1} S_{lm}(c;u) S_{l'm'}(c;u) du \neq \delta_{ll'}\delta_{mm'}$$

Quasi Normal Modes

Imposing the appropriate boundary conditions for the radial solution selects a discrete set of admissible complex frequencies



$\omega_{l,m,n}$

orientation:

prograde $\mathrm{Re}\,\omega > 0$

retrograde $\mathrm{Re}\,\omega < 0$

(for m>0)

$\psi \propto e^{i(m\phi - \omega t)}$

These can be computed using Leaver's continued fraction method (also used to express the angular eigenfunctions as a series in u) [Leaver '85]

Reconstructing the full solution

Expressing the full solution is a non trivial mathematical problem!

$$\psi(t,r,\theta,\phi) = \sum_{l,m,n} \tilde{\psi}_{lmn} e^{-i\omega_{lmn}t} S_{lmn}(\theta) R_{lmn}(r) e^{im\phi}$$

In particular it is not a decomposition over a basis!

In this description, all «modes» are complex exponentials, which would not be the case if we had decomposed on a basis of spheroidal harmonics for a given deformation parameter.

Redefinition of the amplitudes to introduce as a reference time the peak of the 22 and get rid of the radial dependance (we compute our waves at infinity)

$$\psi_4(t,\theta,\phi) = \sum_{l,m,n} \tilde{\psi}_{lmn} e^{-i\omega_{lmn}(t-t_{22})} S_{lmn}(\theta) e^{im\phi}$$

Measuring cosmo. parameters

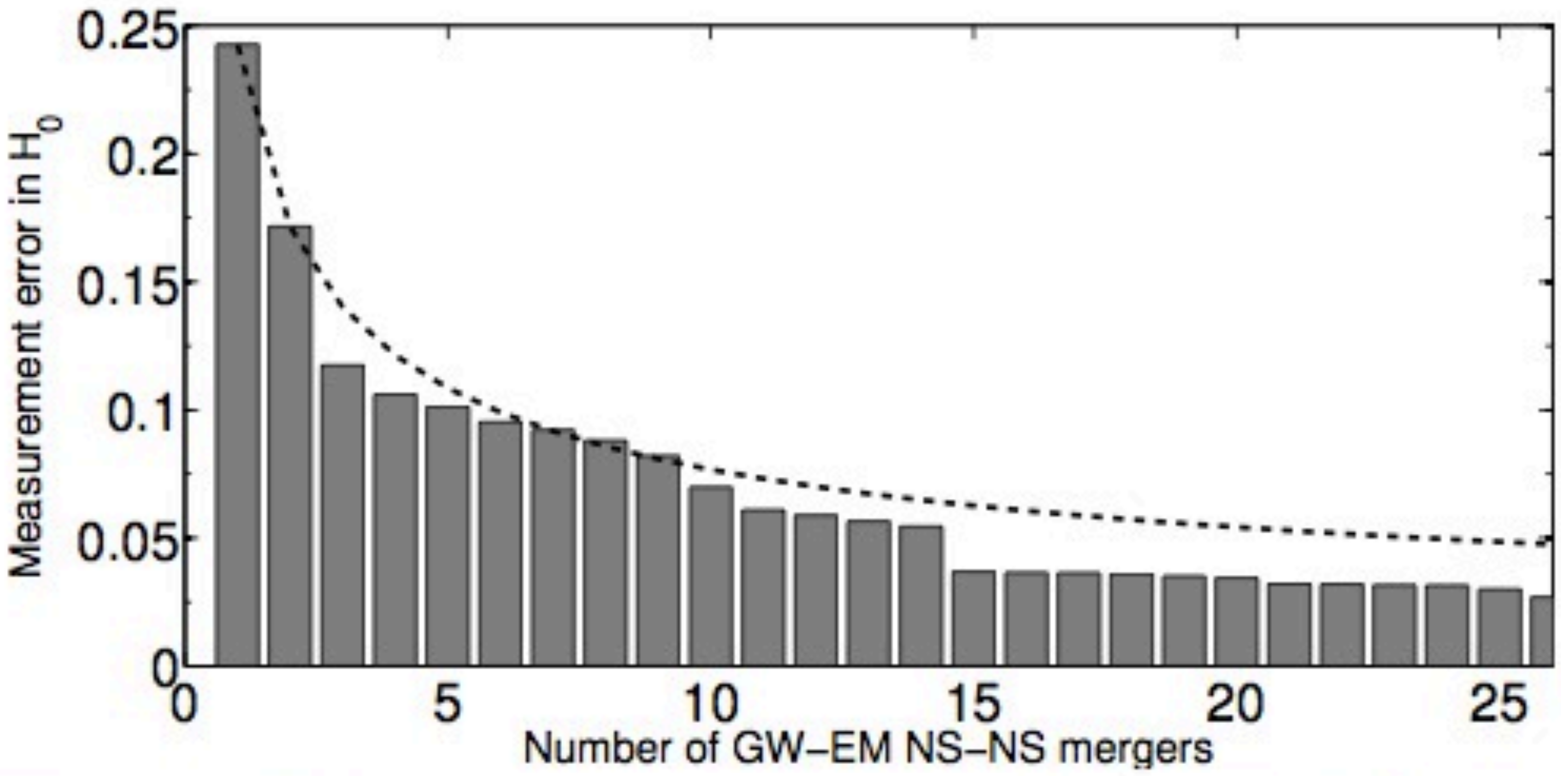


FIG. 2.— H_0 measurement error as a function of the number of multi-messenger (GW+EM) NS-NS merger events observed by a LIGO-Virgo network. The solid bars indicate the 68% c.l. measurement error in H_0 for the *joint* PDF of the independent binary mergers; the dashed line shows the 68% c.l. measurement error in H_0 derived assuming Gaussian errors for each GW-EM merger. When specifying the particular order of events shown, we choose the GW-EM merger in the remaining ensemble with the mean measurement error in H_0.

[Nissanke et al (2009)]

Measuring cosmo. parameters

- Binary neutron stars and black holes are standard sirens (Schutz '86):
  - Distance can be inferred from the gravitational wave signal itself, if (some) information about sky position, orientation

$$h(t) = \frac{\nu M^{5/3}}{D_{\rm eff}} \omega^{2/3} \cos[2\Phi(t-t_0;\, M,\, \nu) + \Phi_0]$$

$$D_{\rm eff} \equiv \frac{D_{\rm L}}{\left[F_+^2(1+\cos^2(\iota))^2 + 4F_\times^2 \cos^2(\iota)\right]^{1/2}},$$

$$\Phi_0 \equiv \Phi_0' + \arctan\left[-\frac{2F_\times \cos(\iota)}{F_+(1+\cos^2(\iota))}\right].$$

(non spinning case)

  - No need for a cosmic distance ladder!
  - Different systematics from SN
  - Errors from lensing (eLISA...)
  - spin breaks degeneracies

Fit to $$D_L = \frac{(1+z)}{H_0} \int_0^z \frac{dz'}{[\Omega_M(1+z')^3 + \Omega_{DE}(1+z)^{3(1+w)}]^{1/2}}$$

- Need to extract redshift:
  - Use electromagnetic counterparts, e.g. Gamma ray bursts [Nissanke et al., arXiv:0904.1017]
  - Assuming a peaked mass distribution [Taylor, Gair, Mandel, arXiv:1108.5161]

Measuring cosmo. parameters

Advanced LIGO/Virgo: H_0 with a few % precision

[Nissanke et al (2009), Del Pozzo (2011)]

Example: With Einstein Telescope

assume 1000 BNS merger over 3 years (1.4 + 1.4) and that H_0 is known

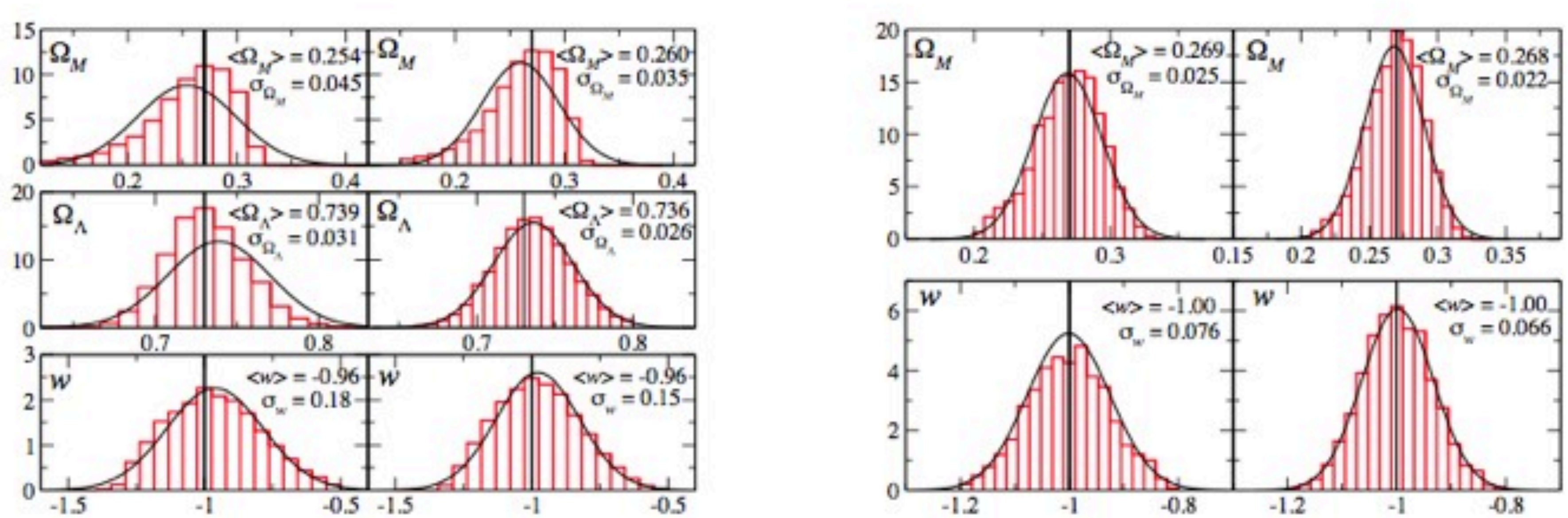


FIG. 2: The plot on the left shows the distribution of errors in $\Omega_{\rm M}$, Ω_Λ and w, obtained by fitting 5,190 realizations of a catalogue of BNS merger events to a cosmological model of the type given in Eq. (1.2), with three free parameters. The fractional 1-σ width of the distributions $\sigma_{\Omega_{\rm M}}/\Omega_{\rm M}$, $\sigma_{\Omega_\Lambda}/\Omega_\Lambda$, and $\sigma_w/|w|$, are 18%, 4.2% and 18% (with weak lensing errors in $D_{\rm L}$, left panels) and 14%, 3.5% and 15% (if weak lensing errors can be corrected, right panels). The plot on the right is the same, but assuming that Ω_Λ is known to be $\Omega_\Lambda = 0.73$, and fitting the "data" to the model with two free parameters. The fractional 1-σ widths in the distribution $\sigma_{\Omega_{\rm M}}/\Omega_{\rm M}$ and and $\sigma_w/|w|$, are 9.4% and 7.6% (with weak lensing errors in $D_{\rm L}$, left panels) and 8.1% and 6.6% (if weak lensing errors can be corrected, right panels).

[Sathyaprakash, Schutz, Van Den Broeck, CQG 2010]