The Upper Critical Dimension of the 3-state Potts Model

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Based on WIP with Ning Su and Zhehan Qin

Why study the 3-state Potts model?

- The critical 3-state Potts model describes phase transitions in nature in $2d$, e.g. ^{4}He atoms on graphite at $\frac{1}{3}$ coverage show second order phase transition with critical exponents that match theory [Alexander '75; Bretz '77; Tejwani et al '80] .
  - In 3d, the Potts lattice model describes cubic ferromagnets with 3 easy axes, e.g. $DyAl_2$, but shows a first order phase transition [Mukamel et al '76; Barbara et al '78] (so NOT critical).
- Simplest QFT after the Ising model, i.e. few relevant operators, global symmetry just S_3.
- The critical and tricritical q-state Potts models are believed to demonstrate merger and annihilation scenario for critical points, as function of either q or d near $q = 3$.

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Merger and annihiliation scenario

- Consider two families of unitary CFTs parameterized by s with same global symmetry and number of relevant operators, except one CFT has extra relevant singlet.
- As s changes, the CFT data of these two CFTs gets closer until a s_{crit}, where the two CFTs become identical and go off into the complex plane (no longer unitary) [Kaplan, Lee, Son 09'] .
  - The "extra" relevant singlet operator becomes marginal at s_{crit}.
- Critical and tricritical q-state Potts have same S_q global symmetry, but tricritical has extra relevant singlet operator for $q = 3$.
- In 2d, critical and tricritical q-state Potts let $s = q$: as $q \to q_{\text{crit}} = 4$ the theories merge [Neinhuis, Berker, Riedel, Schick '79] and then go off into complex plane [Gorbenko, Rychkov, Zan '18] .

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q_{crit} for critical Potts for $d > 2$

- In $d \geq 4$, proven that $q_{\text{crit}} = 2$ [Aharony, Pytte '81] .
  - No merger/annihilation in this case, instead $q = 2$ critical Potts (i.e. Ising) becomes free at $d = 4$.
- In 3d, lattice Monte Carlo suggests $q_{\text{crit}} \sim 2.45$ [Lee, Kosterlitz '91] .
- In $2 < d \leq 3$, various estimates of $(d_{\text{crit}}, q_{\text{crit}})$:
  - Lattice Monte Carlo of generalization of Potts model gives $(2.5, 2.68)$ [Barkema, de Boer '91] .
  - $d = 4 - \epsilon$ and $q = 2 + \epsilon$ expansion gives $(4 - \epsilon, 2 + \epsilon) + O(\epsilon^2)$ [Aharony, Pytte '81] see also [Newman, Riedel, Mutto '83] .
  - RG analysis gives $(2.32, 2.85)$ [Nienhuis, Riedel, Schick '80] .
- This talk: Use bootstrap to find upper critical dimension $d_{\text{crit}} \sim 2.5$ for $q = 3$ via merger/annihilation of critical/tricritical CFTs.

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Outline of talk

- Define q-state Potts model in any spacetime d, and the critical and tricritical fixed points.
- Review exact solutions in 2d for various q.
- In 2d, use conformal bootstrap to find kinks that correspond to the exact solutions of the 3-state critical and tricritical Potts CFTs.
- Using same bootstrap setup, increase d and find that critical and tricritical kinks merge and disappear around $d \sim 2.5$.

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Lattice definition of Potts for any d, q

- Consider d-dimensional square lattice of random spins with Hamiltonian for $s_i \in \{1, 2, \ldots, q\}$ [Potts '52] :

$$Z = \sum_{\{s_i\}} e^{-H[\{s_i\}]}, \qquad H[\{s_i\}] = \beta \sum_{\langle ij \rangle} \delta_{s_i, s_j}$$

- Has exact S_q global symmetry.
- At large β ordered phase with q degenerate ground states with S_q broken and 1 spin value prefered, at small β have disordered phase with one ground state with S_q symmetry.
- Tune $\beta = \beta_{\text{crit}}$ to get phase transition called critical Potts model.
- Consider dilute lattice model where some lattice sites are vacant, can tune β to get same critical Potts model, can tune both β and chemical potential of vacancies to get tricritical Potts model.

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$q = 2$ Potts (Ising)

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  - In $d = 2$ Z_2 even subsector of tricritical has enhanced superconformal algebra, but not for $d > 2$ (tricritical in $d > 2$ unrelated to $\mathcal{N} = 1$ super-Ising model).
- Critical Ising has two relevant operators (1 $\mathbb{Z}_2$ odd, 1 $\mathbb{Z}_2$ even), tricritical has 4 relevant operator (2 $\mathbb{Z}_2$ odd, 2 $\mathbb{Z}_2$ even).
- In 2d, critical and tricritical are lowest two unitary diagonal minimal models $M_{4,3}$ ($c = \frac{1}{2}$) and $M_{5,4}$ ($c = \frac{7}{10}$), i.e. exactly solvable
- Diagonal minimal models $M_{p+1,p}$ described by Lagrangian $(\partial\phi)^2 + \lambda(\phi^2)^{p-1}$, has upper critical dimension $d_{\text{crit}} = \frac{2p-2}{p-2}$ when $(\phi^2)^{p-1}$ becomes marginal $\Leftrightarrow$ theory becomes free.
  - Thus $d_{\text{crit}} = 4$ for critical Ising, $d_{\text{crit}} = 3$ for tricritical Ising, no merger/annihilation.

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- Diagonal minimal models $M_{p+1,p}$ described by Lagrangian $(\partial\phi)^2 + \lambda(\phi^2)^{p-1}$, has upper critical dimension $d_{\text{crit}} = \frac{2p-2}{p-2}$ when $(\phi^2)^{p-1}$ becomes marginal $\Leftrightarrow$ theory becomes free.
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Other $q \neq 3$ Potts

- For $q = 4$, tricritical and critical Potts are the same unitary CFT: free scalar compactified on on $S_1/\mathbb{Z}_2$ with radius $R = 1/\sqrt{2}$ with three marginal operators [Dijkgraaf, Verlinde^2] .
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$q = 3$ Potts CFT: General definitions

- Define 3-state Potts CFTs in general d as CFT with S_3 global symmetry and certain number of relevant operators.
- S_3 has 3 irreps: singlet $\mathbf{0}$, sign $\mathbf{0}_-$ (odd under $\mathbb{Z}_2 \subset S_3$), and charged $\mathbf{1}$ (± 1 charge under $\mathbb{Z}_3 \subset S_3$).
- Critical Potts has two relevant charged operators σ, σ', and one relevant singlet ϵ.
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- Critical and tricritical Potts in 2d are minimal models $M_{6,5}$ ($c = \frac{4}{5}$) and $M_{7,6}$ ($c = \frac{6}{7}$) with non-diagonal modular partition functions.
  - Recall that diagonal $M_{6,5}$ and $M_{7,6}$ are $\mathbb{Z}_2$ invariant $(\phi^2)^4$ and $(\phi^2)^5$ CFTs, multi-critical generalizations of Ising model.
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Primaries in 2d critical Potts

- Virasoro primaries (with integer spin) given by subset of $M_{6,5}$ that appear in non-diagonal torus partition function labeled by $(\Delta, j, \mathbf{r})$ for scaling dimension Δ, Lorentz spin j, and S_3 irrep $\mathbf{r}$:

$$\sigma = (2/15\,, 0\,, \mathbf{1})\,, \qquad \sigma' = (4/3\,, 0\,, \mathbf{1})\,,$$
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- Conserved current W is the generator of $W(3)$ algebra.
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- OPE coefficients in minimal models are all computable in principle, for non-diagonal models method was outlined in [Fuchs, Klemm '89] .
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$$\lambda^{\epsilon}_{\sigma\bar{\sigma}} = \frac{\sqrt{\frac{1}{2}(1+\sqrt{5})}\Gamma(\frac{3}{5})^2}{2\Gamma(\frac{2}{5})\Gamma(\frac{4}{5})}$$

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- OPE coefficients in minimal models are all computable in principle, for non-diagonal models method was outlined in [Fuchs, Klemm '89] .
- This algorithm was carried out for the critical Potts in [McCabe, Wydro '95] , see also [Migliaccio, Ribault '18] . E.g.:

$$\lambda^{\epsilon}_{\sigma\bar{\sigma}} = \frac{\sqrt{\frac{1}{2}(1+\sqrt{5})}\Gamma(\frac{3}{5})^2}{2\Gamma(\frac{2}{5})\Gamma(\frac{4}{5})}$$

- For tricritical Potts, this algorithm not done yet, but a few OPE coefficients were computed in [Zamolodchikov, Fateev '87] .
- In both critical and tricritical Potts, many OPE coefficients that would be allowed by S_3 symmetry in general $d > 2$ are zero in 2d due to enhanced Virasoro (and W-algebra) constraints. E.g.:

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$W(3)$ minimal models [Zamolodchikov, Fateev '86]

- Recall that critical Potts ($c = \frac{4}{5}$) is lowest member of family of $W(3)$ minimal models with $c = 2(1 - \frac{12}{p(p+1)})$ for $p \geq 4$.

- Scalar Virasoro primaries $\Phi \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}$ for $n_1 + n_2 \leq p - 1$ and $m_1 + m_2 \leq p$ labeled by Dynkin labels of $\mathfrak{sl}(3) \times \mathfrak{sl}(3) \subset W(3) \times \overline{W(3)}$ with known $\Delta \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}$.

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- Expand σ correlator in blocks for each irrep in $\mathbf{1} \otimes \mathbf{1} = \mathbf{0} \oplus \mathbf{0}_- \oplus \mathbf{1}$:

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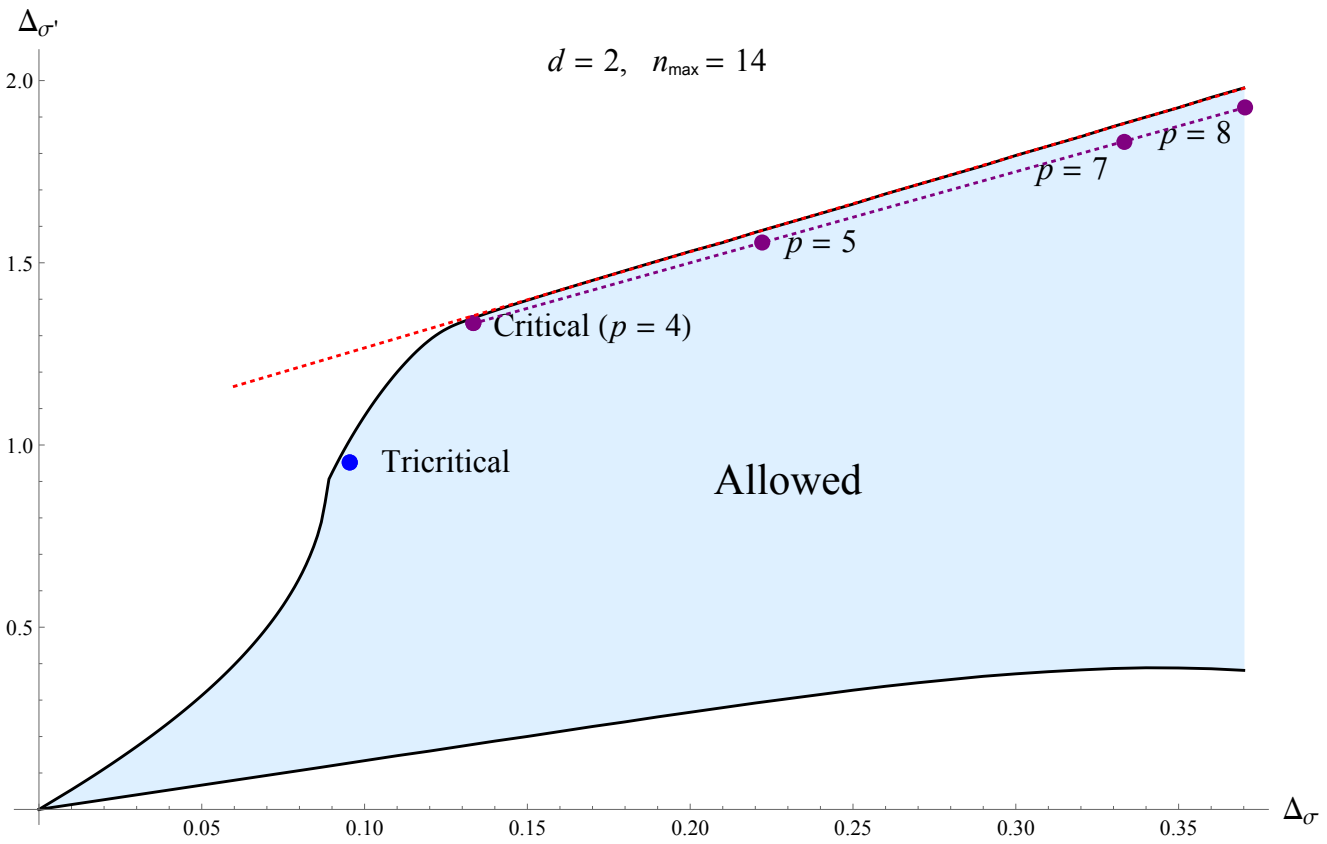

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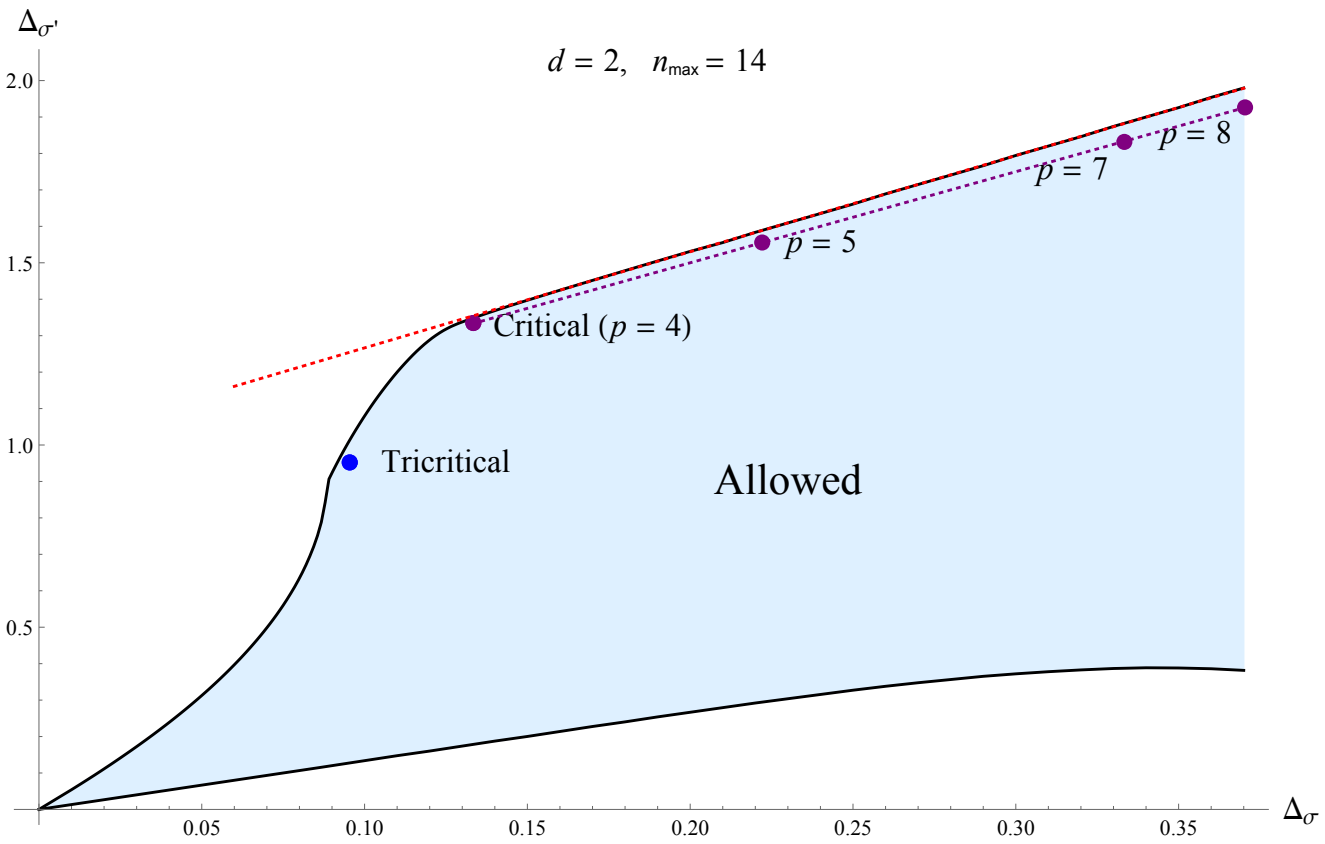

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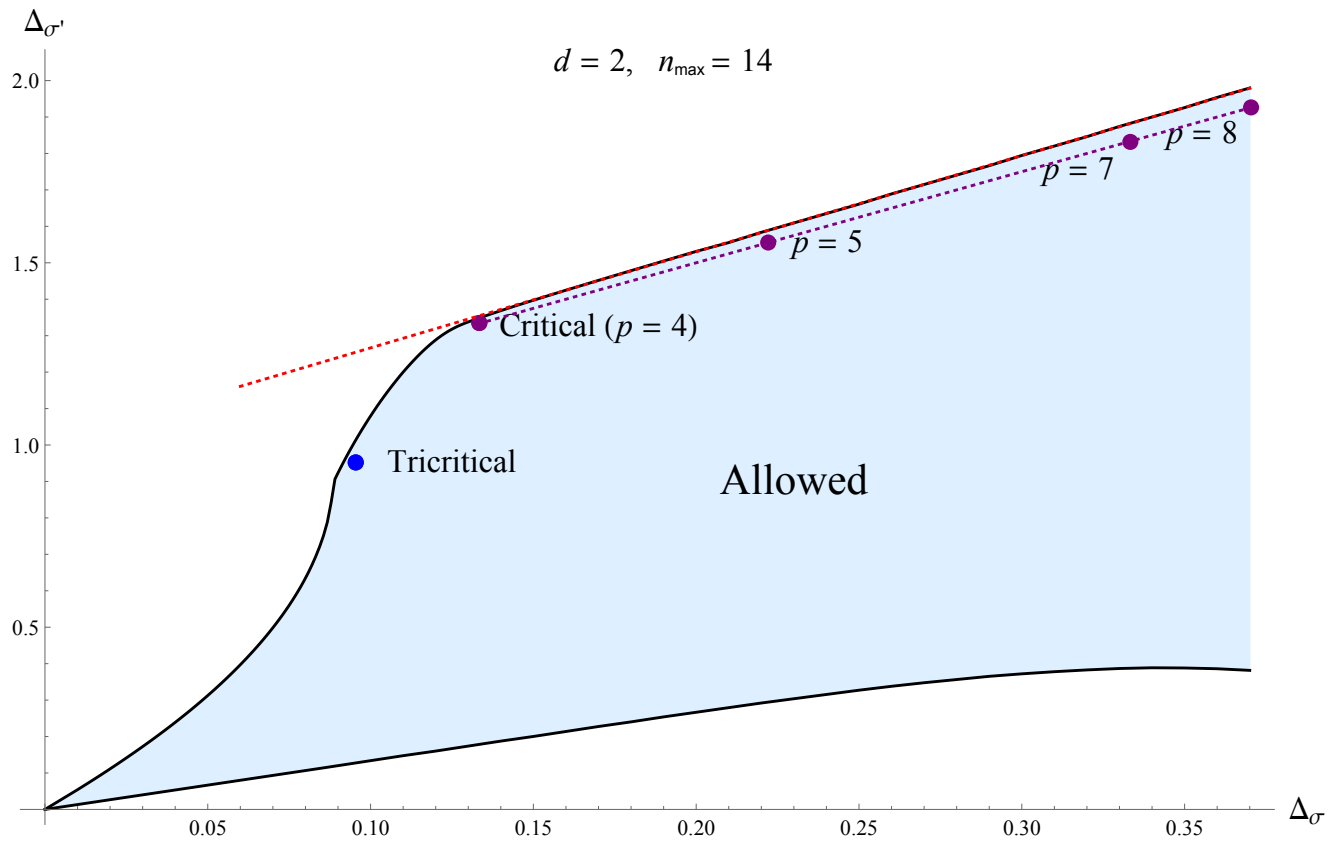

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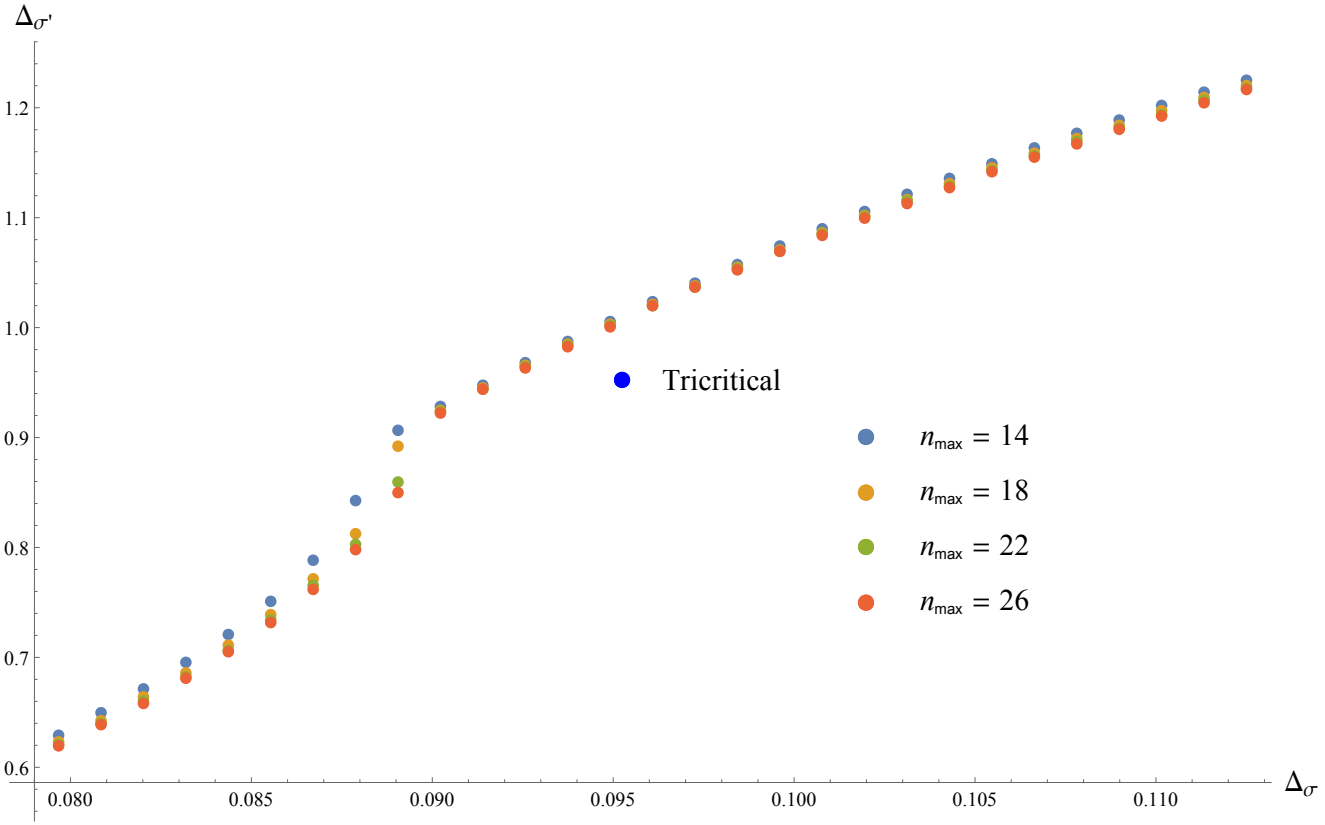

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- Note that plot is very zoomed in, kink still matches expected point to few percent error.

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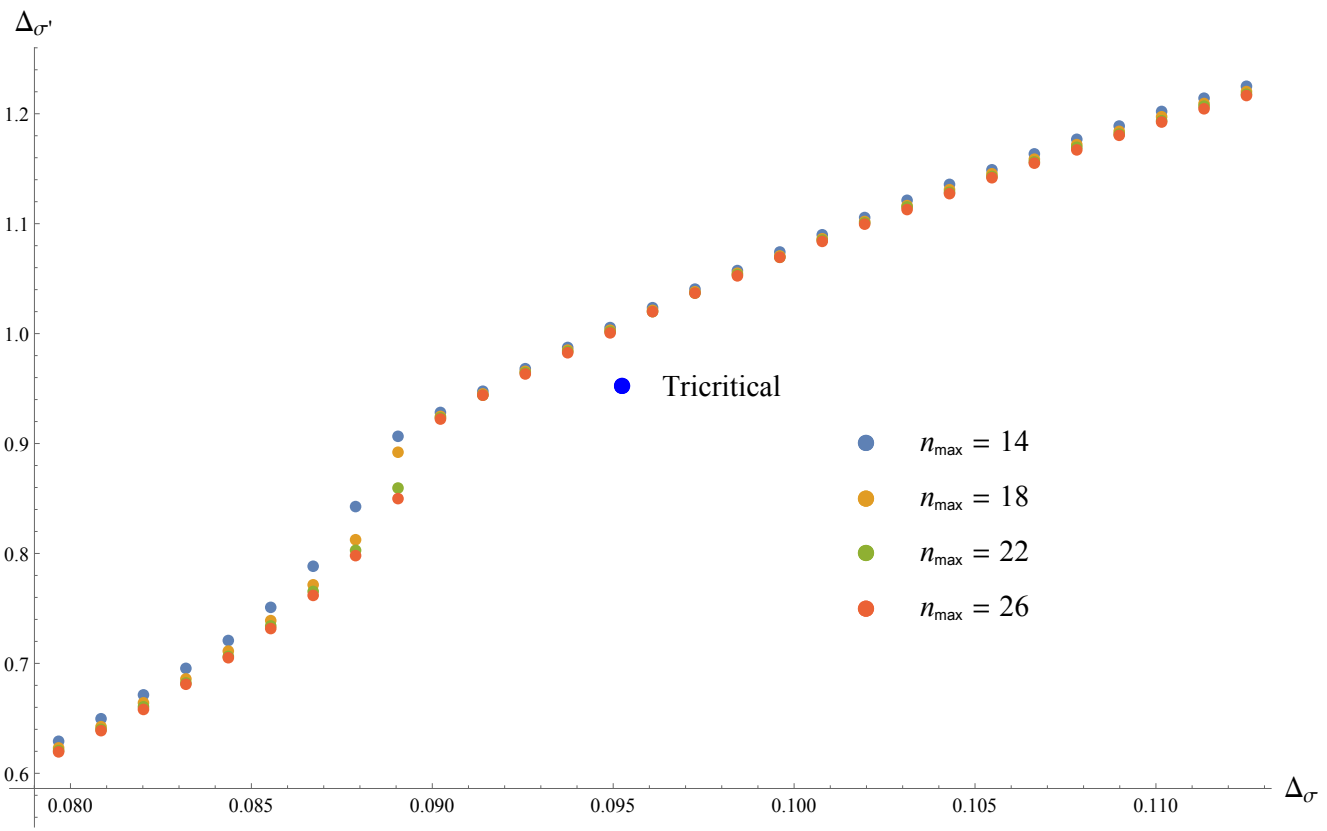

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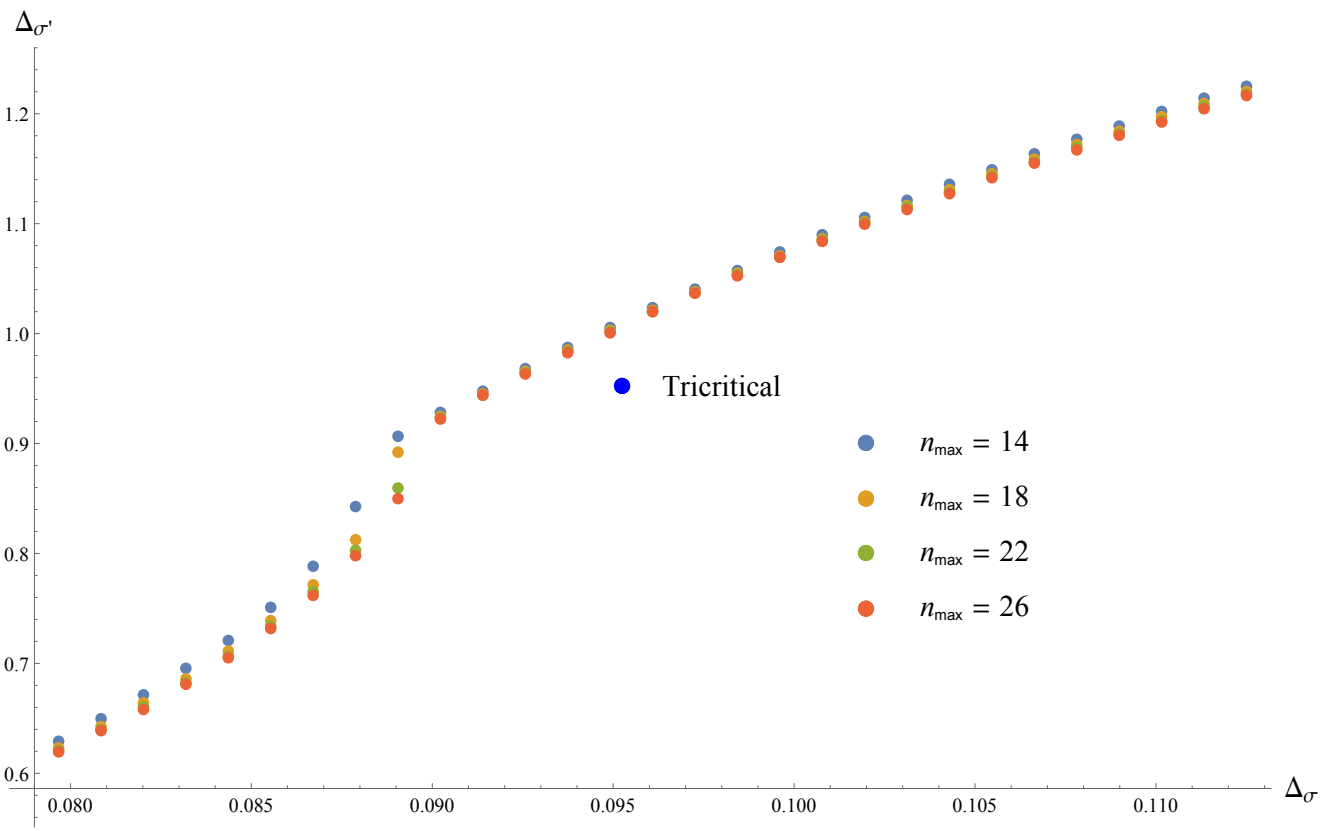

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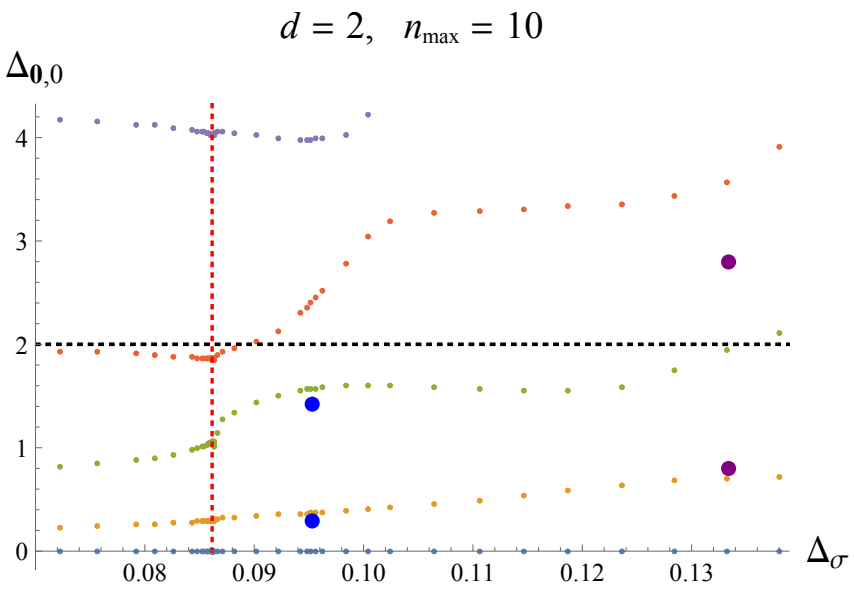
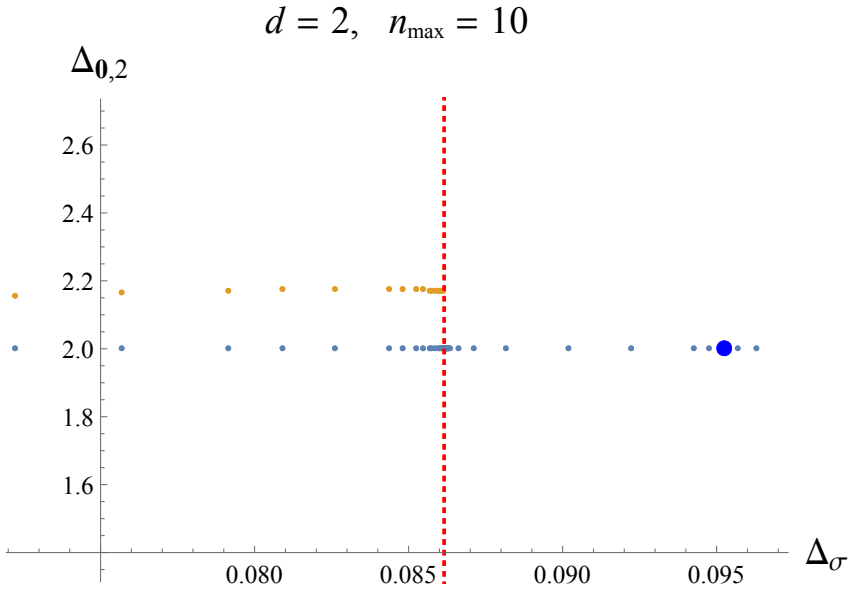
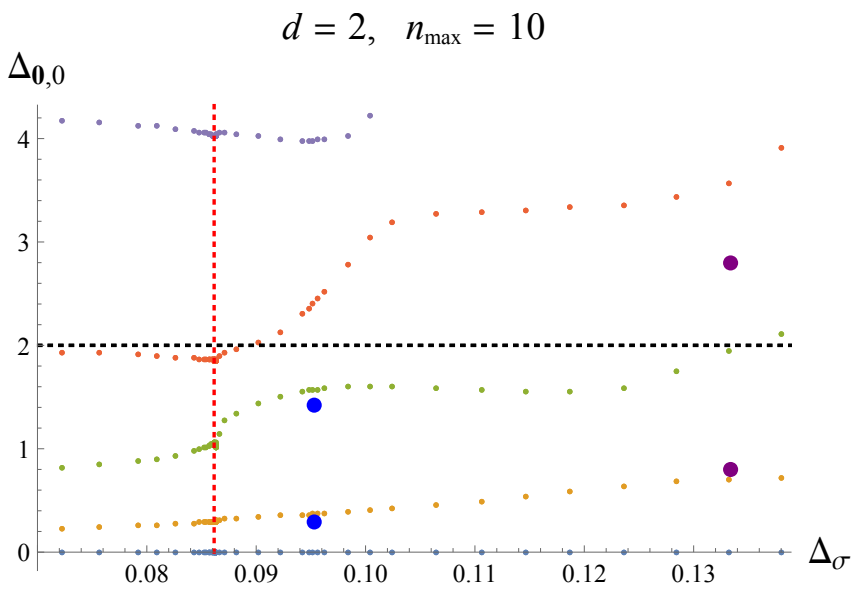
Bootstrap spectrum in 2d (singlet sector)

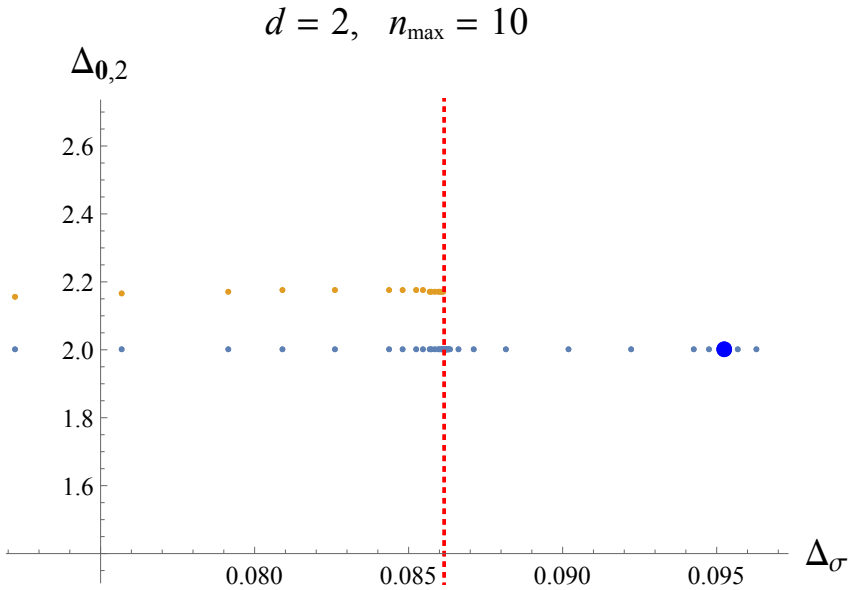




- Spectrum from boundary of allowed region, made with navigator function [Reehorst, van Rees, Rychkov, Sirois, DSD, Su '21]
- Red dotted line shows where kink is, see operator rearrangement in spin 2 singlet channel, black dotted line is where operators would cross marginality, blue/purple dots denote exact spectrum for tricritical/critical.
- Sometimes fake operators show up in spectrum at unitarity bound.

Bootstrap spectrum in 2d (singlet sector)





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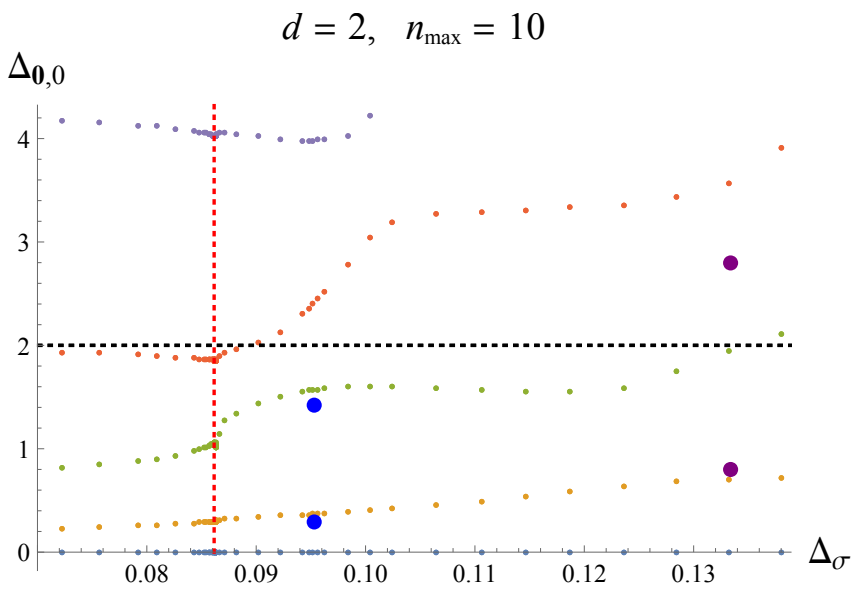

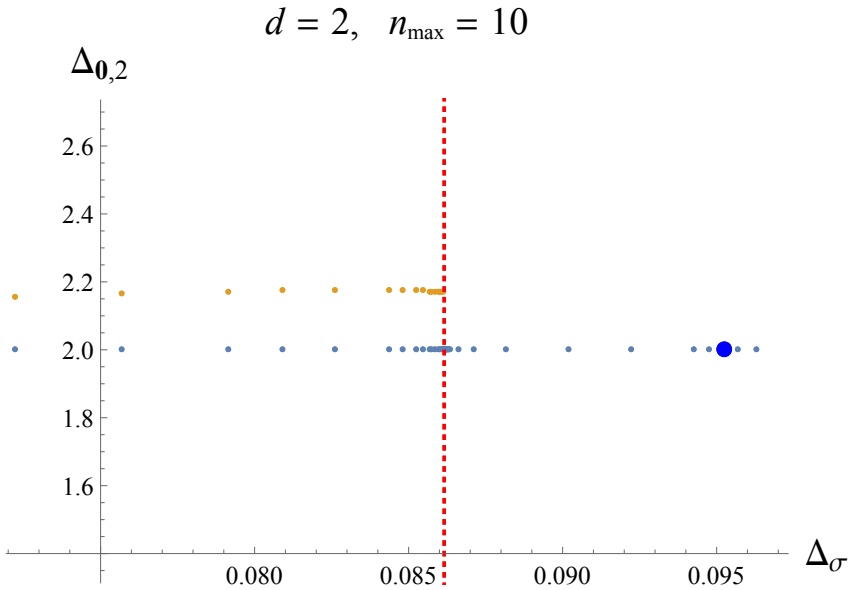

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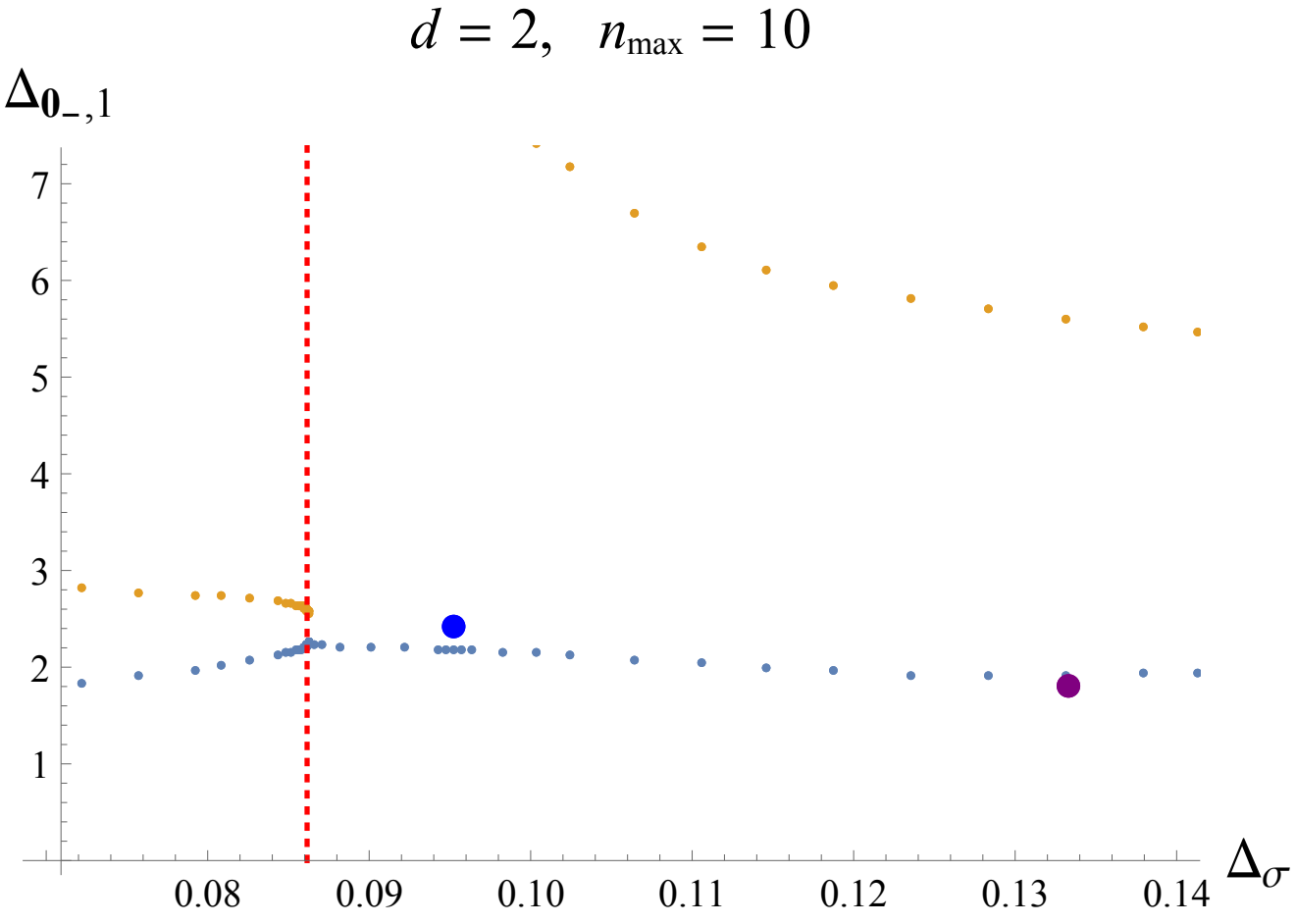

- Approximate match with exact value in blue/purple for tricritical/critical.
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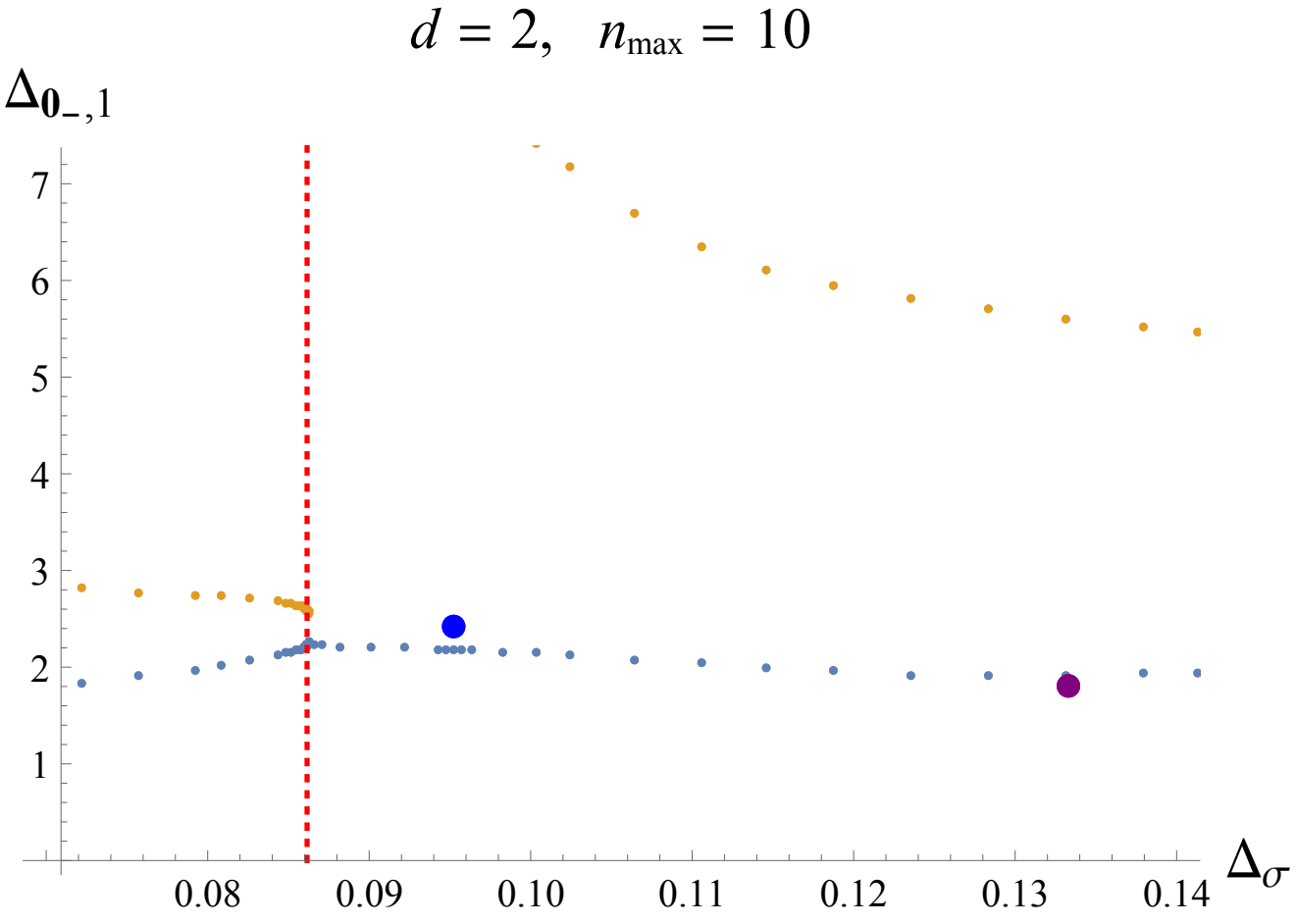

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Bootstrapping in fractional d

- Our bootstrap changes in two ways as we go to $d > 2$:
  - We use global conformal blocks in $SO(d+1,1)$ (smooth functions of d [Dolan, Osborn '03]) as we did in 2d.
  - We impose a gap to d in the scalar **1** sector, and insert two relevant operators (as in 2d).
- Some CFTs in fractional d have operators with large Δ that violate unitarity, e.g. Ising model [Hogervorst, Rychkov, van Rees '16] .
  - Contribution of high dimension operators highly suppressed in block expansion, so hard to see from numerical bootstrap, which matches ϵ expansion and interpolates between known results in integer dimensions. [El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi '14; Cappelli, Maffi, Okuda '18] .
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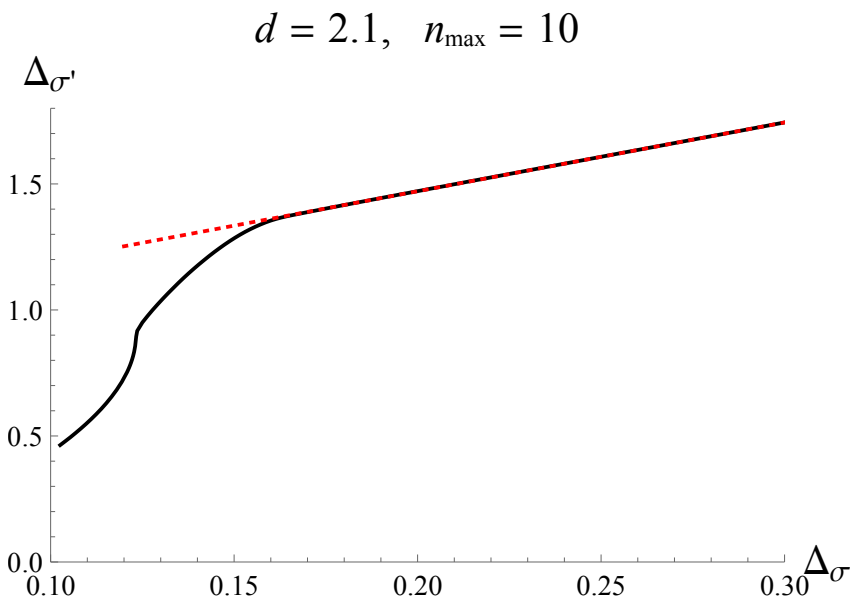
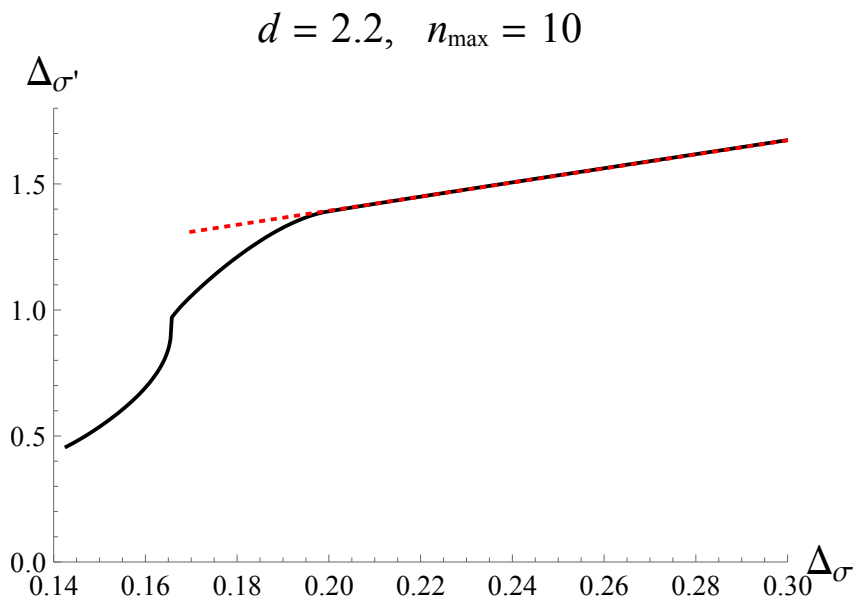
Potts plots in $d > 2$





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- Still see clear tricritical kink, getting closer to critical kink (where line starts to curve).

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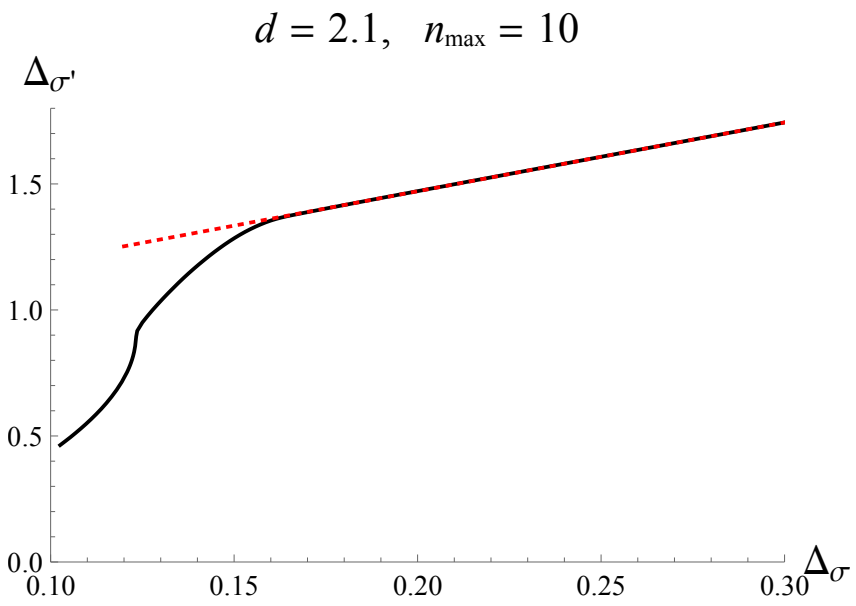

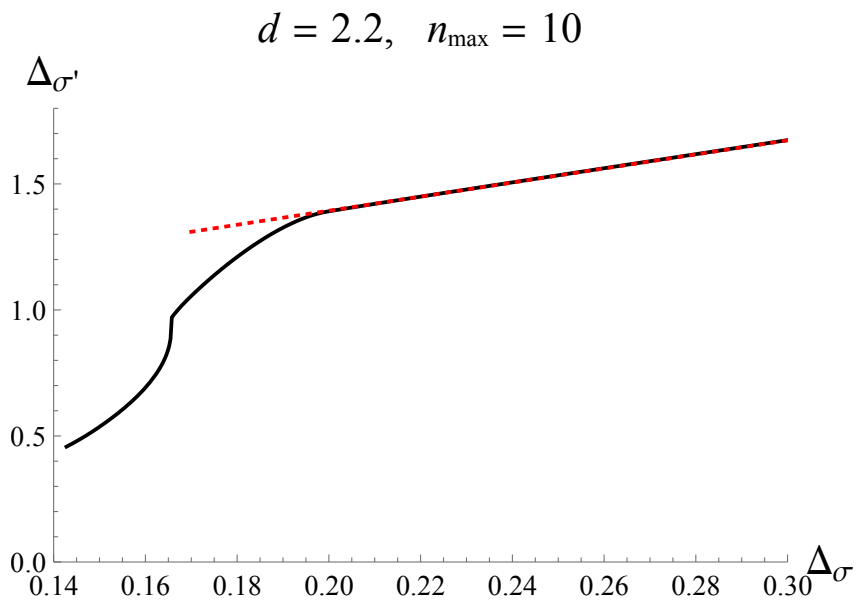

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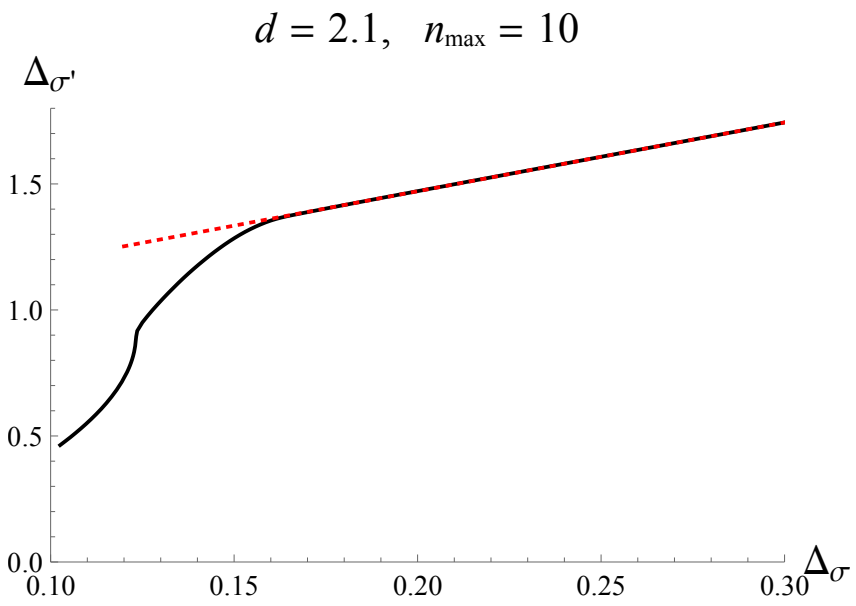

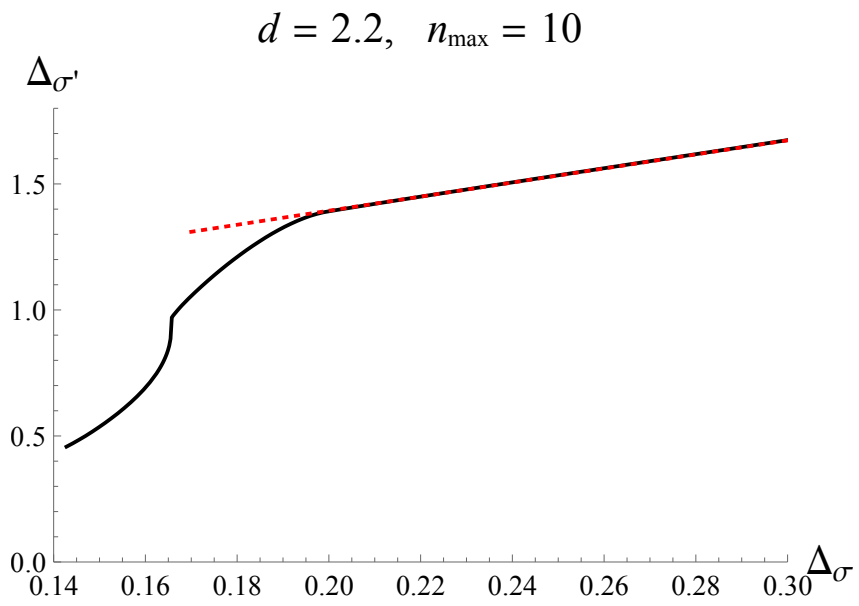

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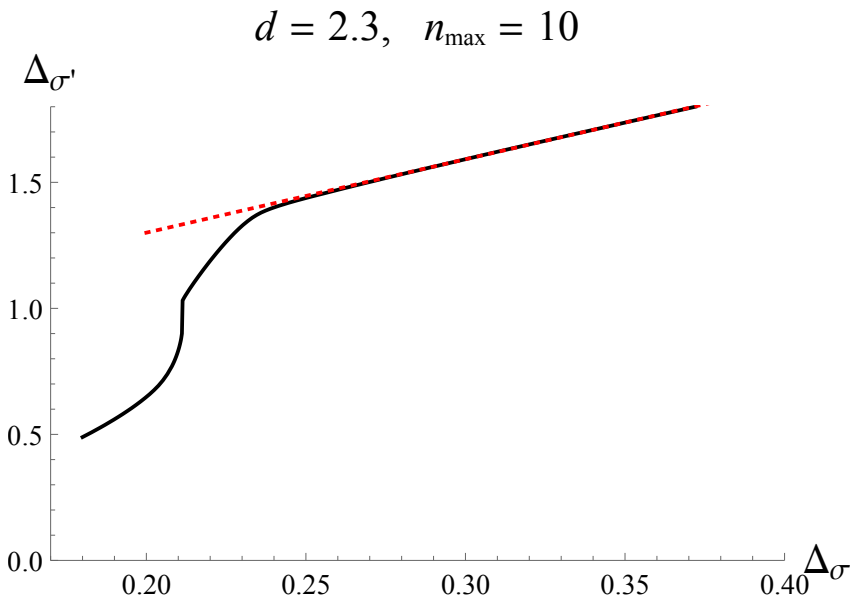

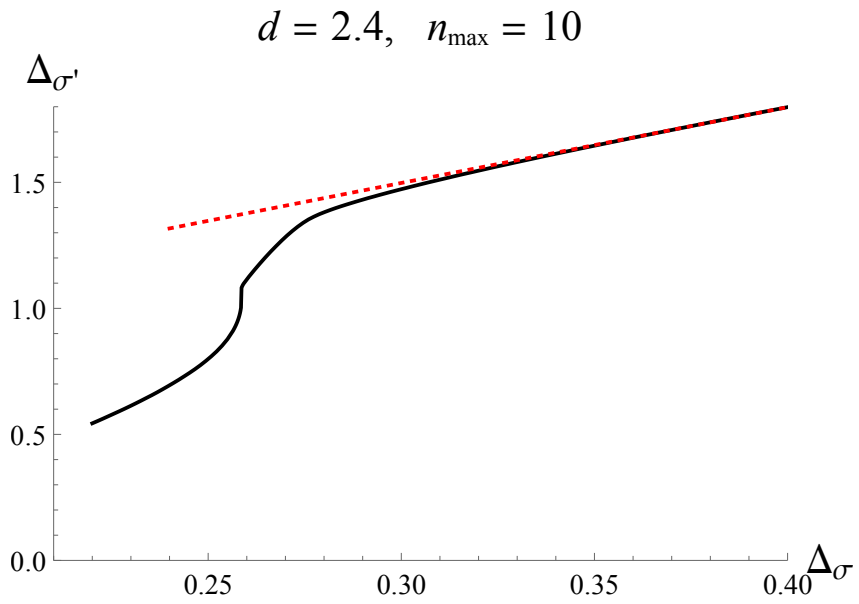

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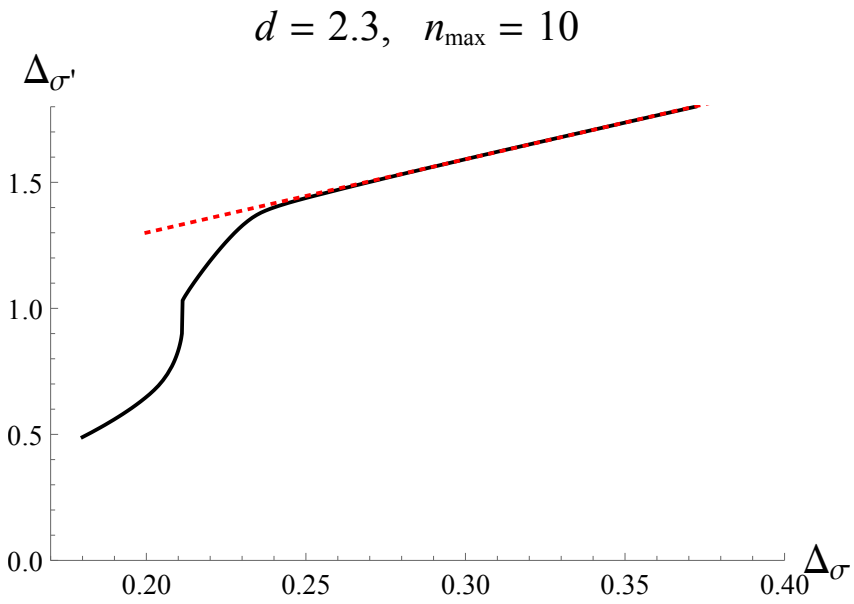

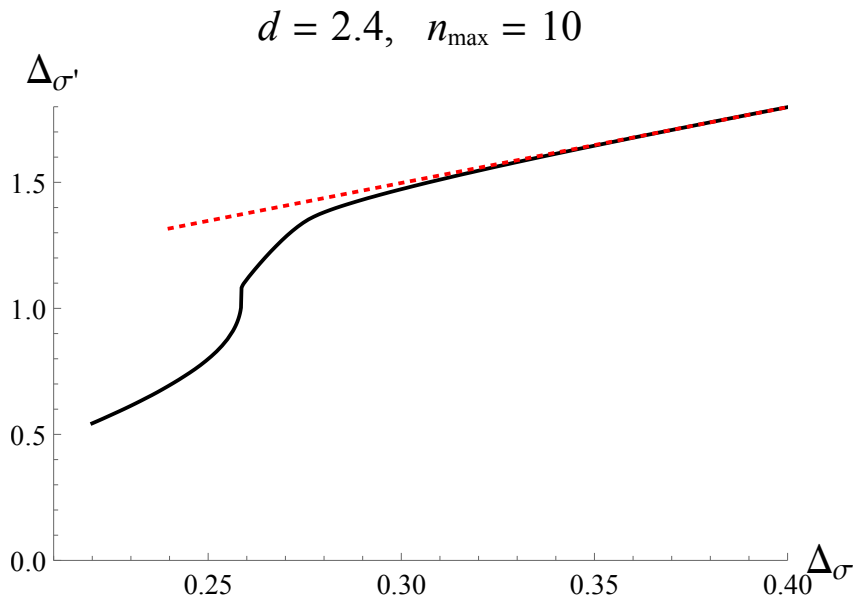

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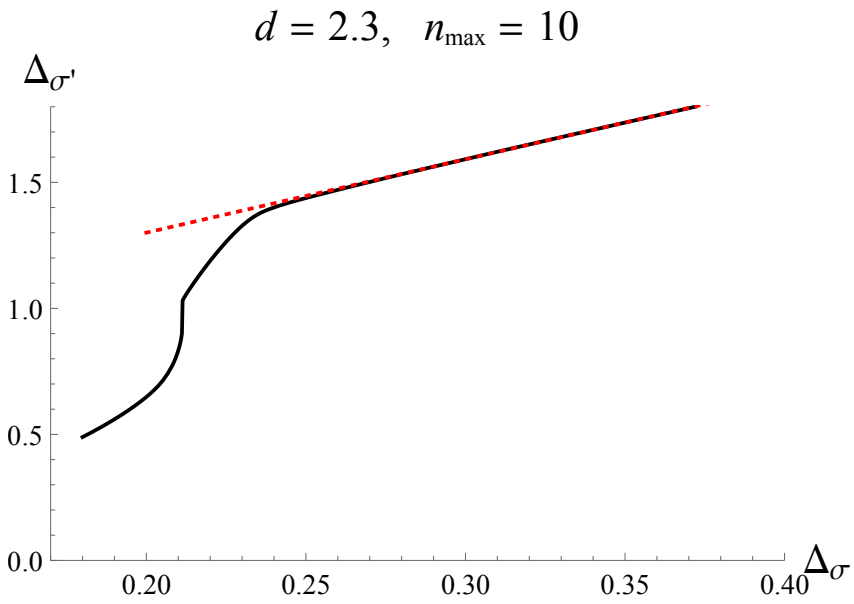

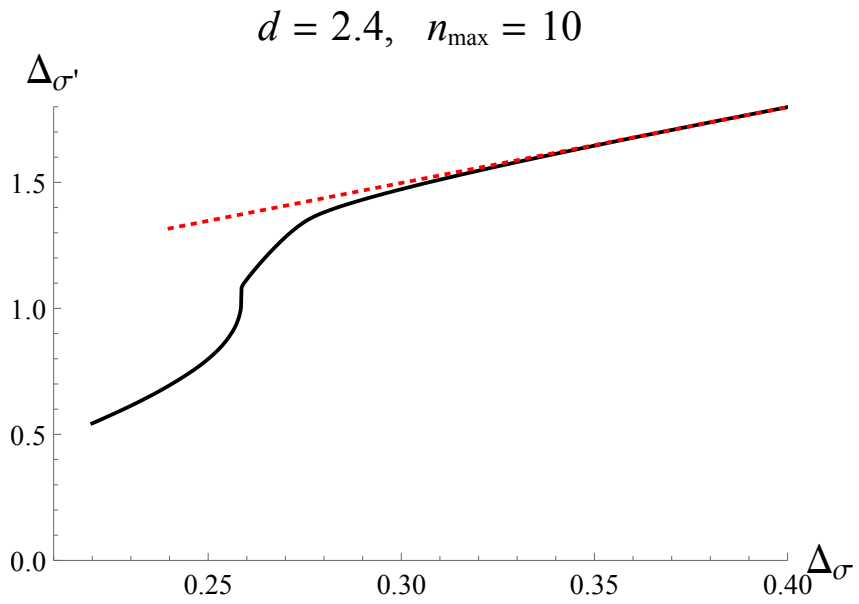

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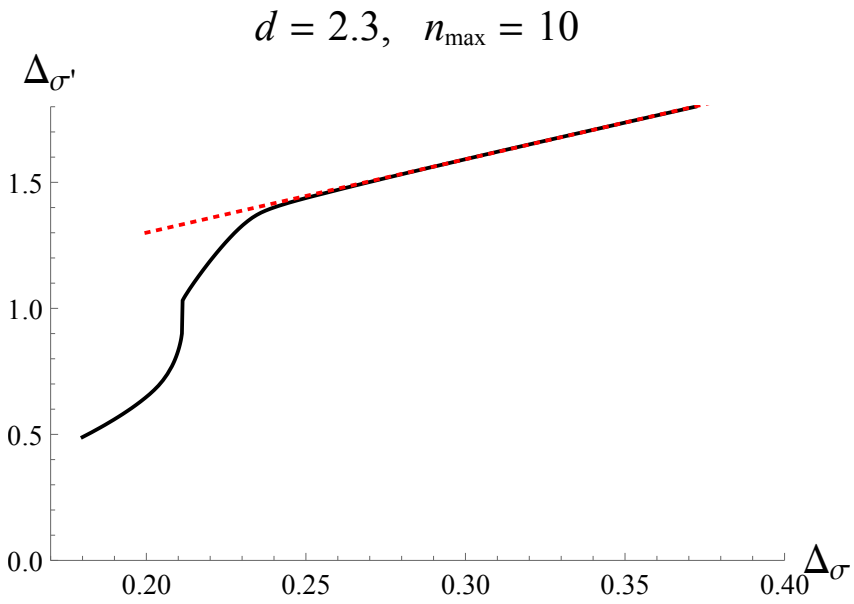

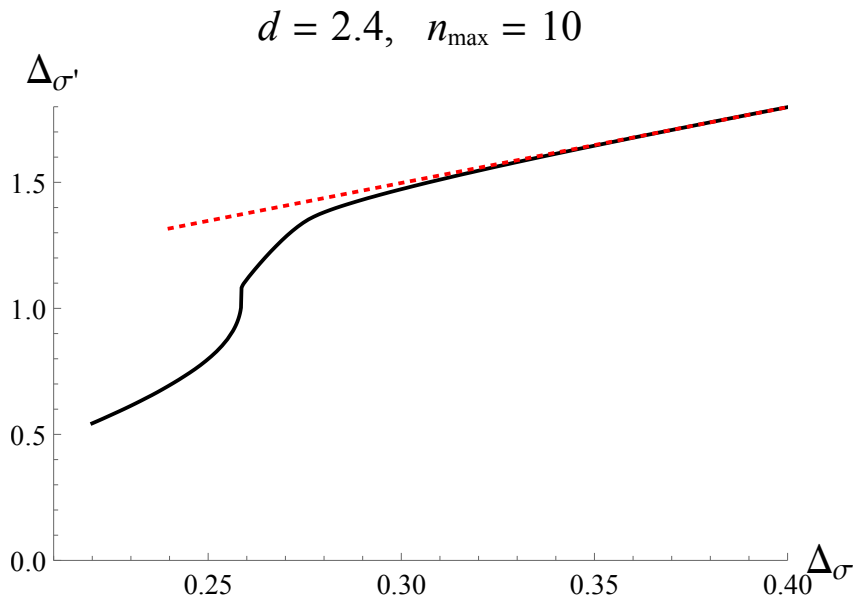

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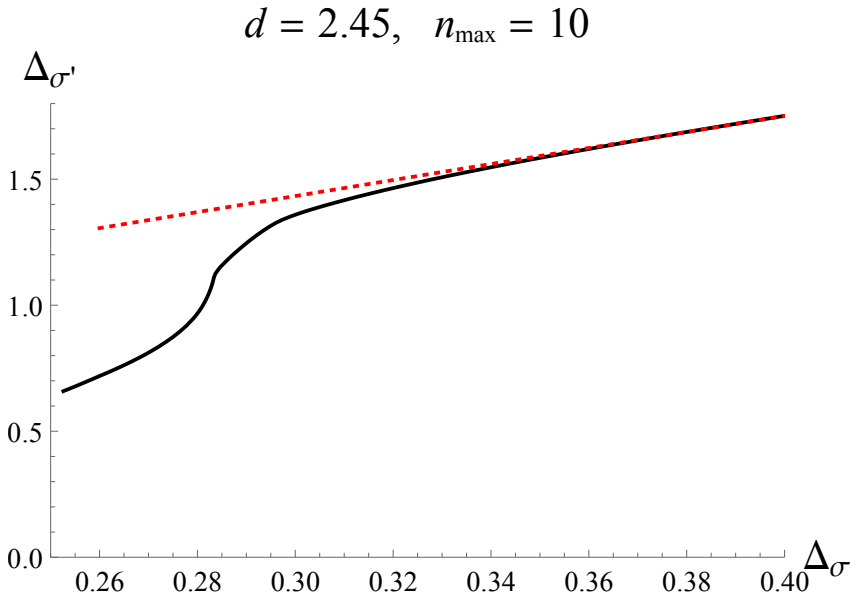
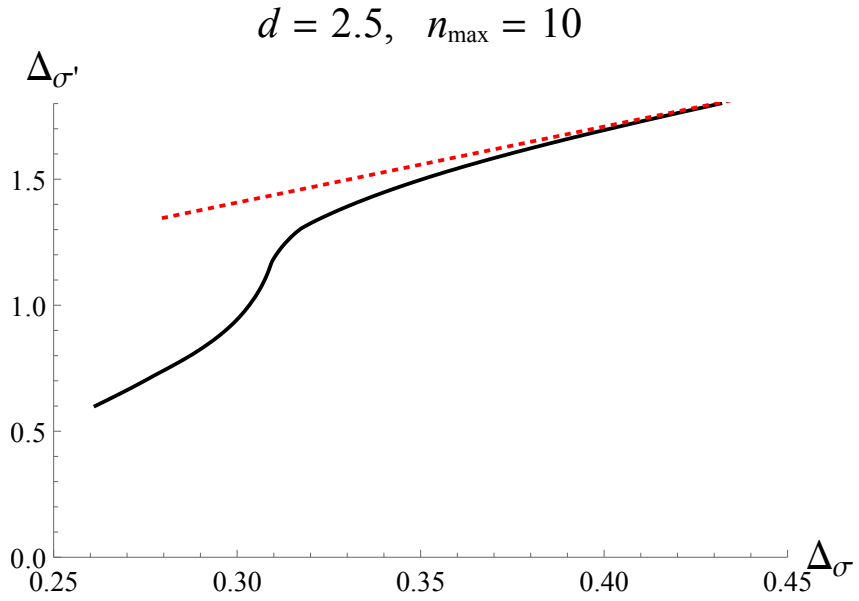
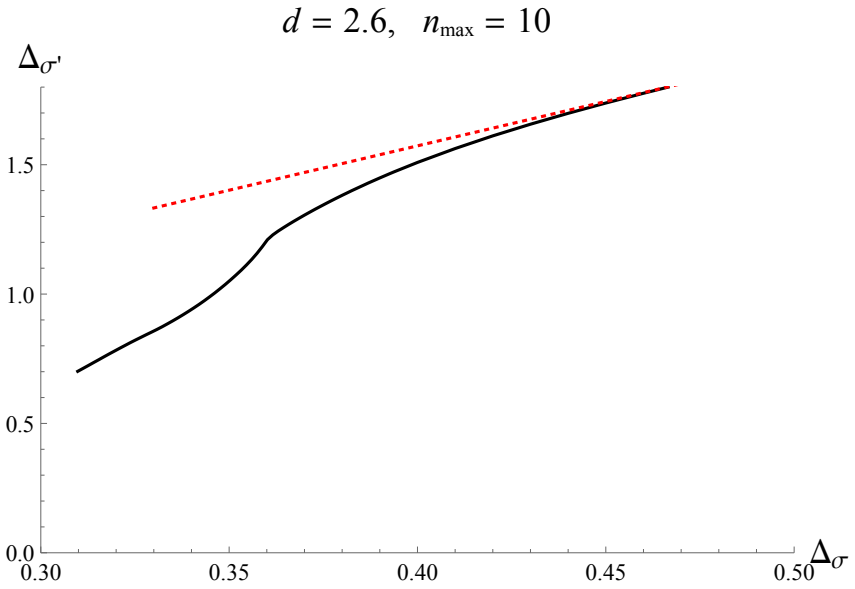
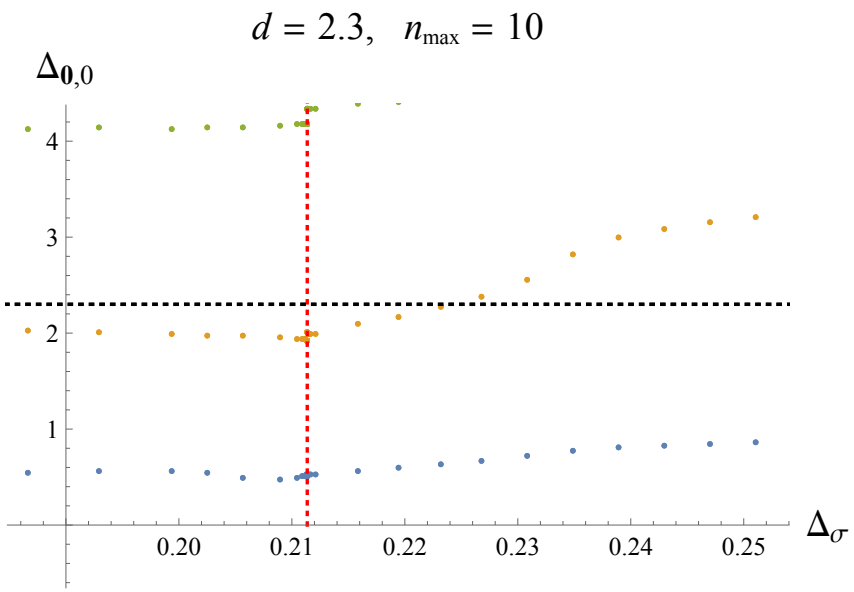
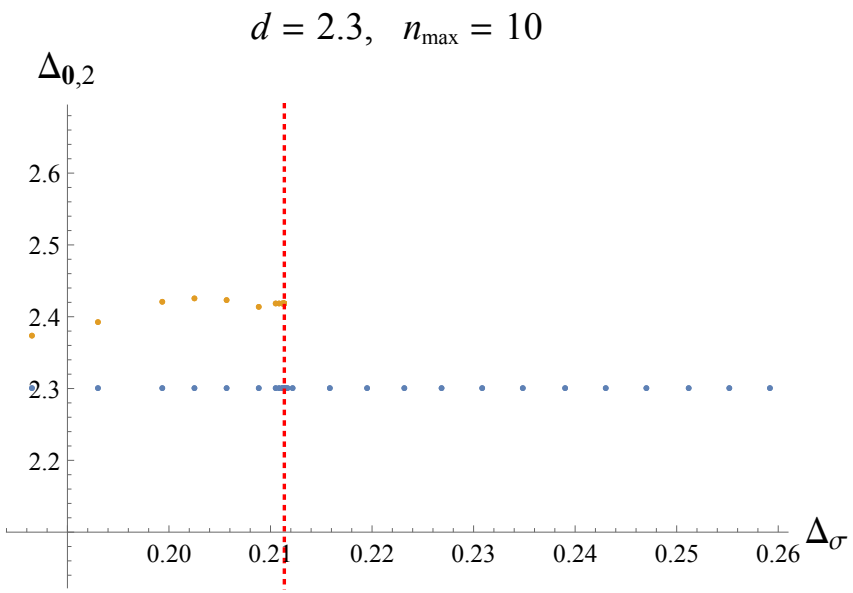






Bootstrap spectrum in $d = 2.3$ (singlet sector)





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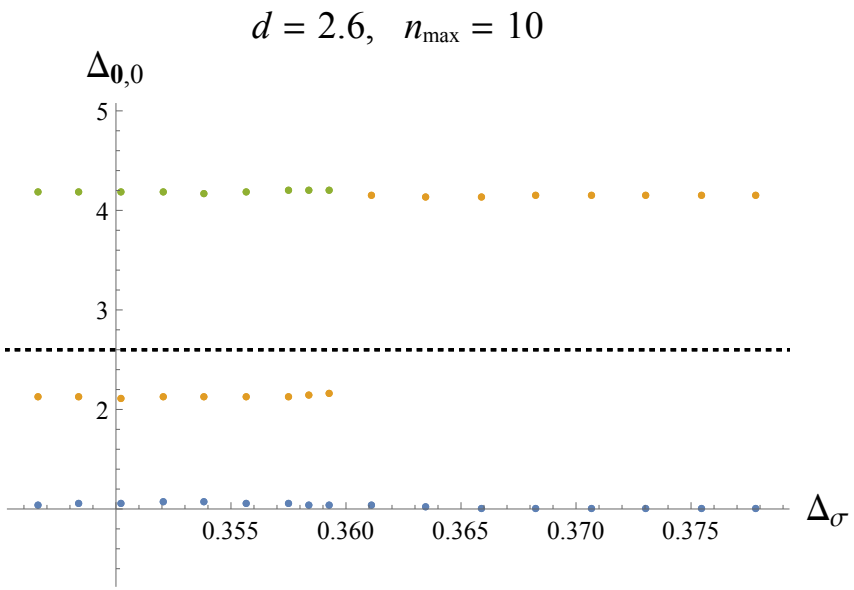
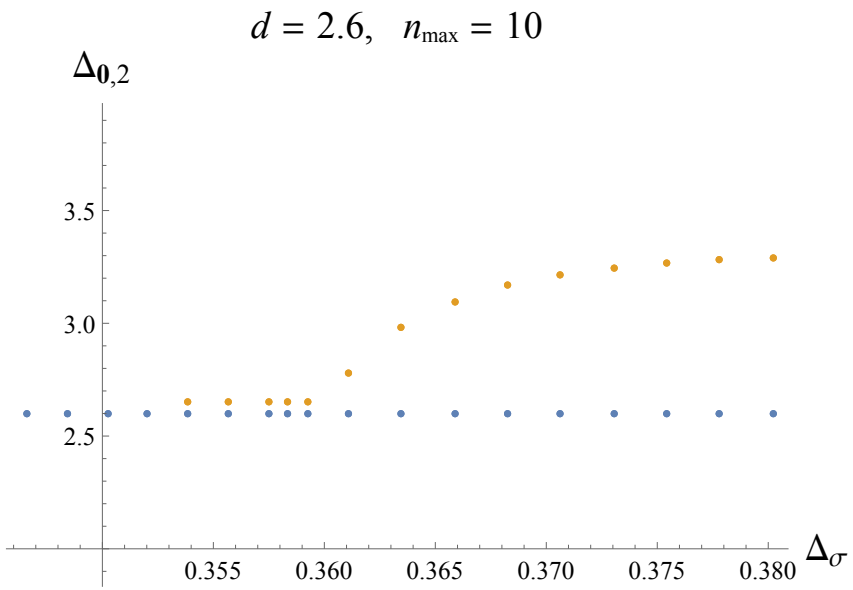
Bootstrap spectrum in $d = 2.6$ (singlet sector)





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- We also no longer see the operator near marginality in the spin 0 channel.

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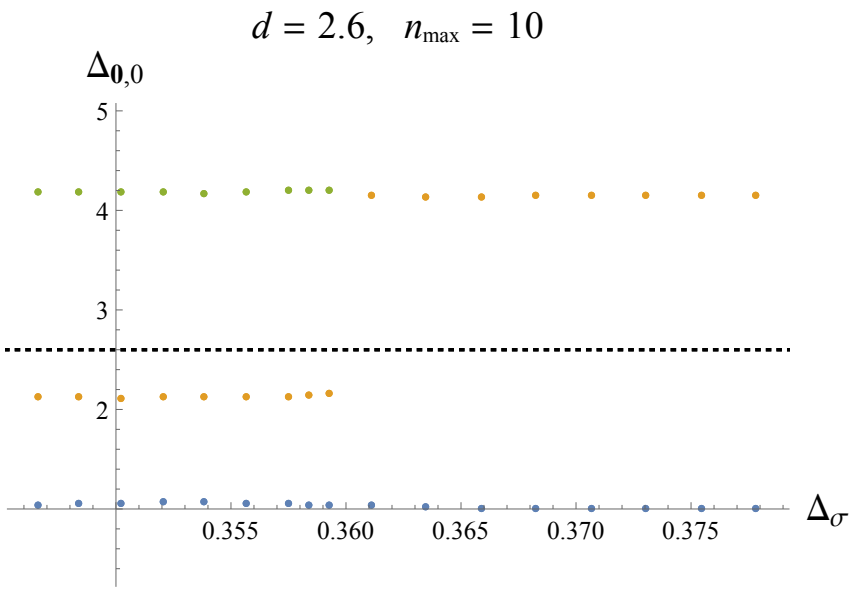

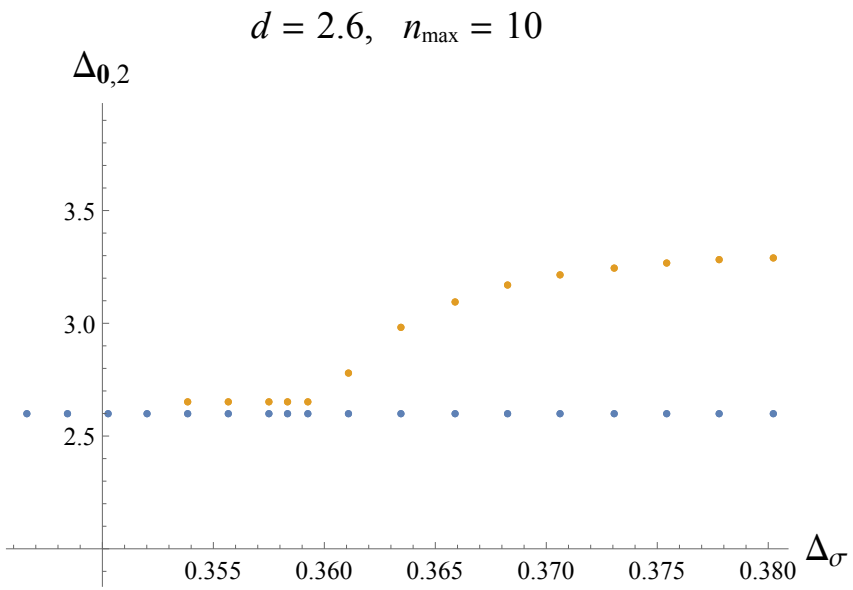

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Other correlators?

- With just σ single correlator, can plot σ versus ϵ with only a gap on scalar **0** for critical Potts. Gives big allowed region, nothing interesting (like 2d Ising model numerical bootstrap).
  - To do: 3d plot of $\sigma, \sigma', \epsilon$ for critical Potts imposing all gaps, or 4d plot of $\sigma, \sigma', \epsilon, \epsilon'$ for tricritical, see if kinks become sharper.
- Mixed correlators of $\sigma, \sigma', \epsilon$ with gaps in both sectors does NOT gives islands (without additional assumptions that are not justified for $d > 2$).
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- Found less sharp feature at expected critical Potts in 2d (previously observed by [Rong, Su '17]), where straight line saturated by $W(3)$ minimal models starts to curve.
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- Bootstrap c_T, see if it shows minimum at Potts theories, like for Ising model [El-Showk, Paulos, Poland, Rychkov, DSD, Vichi '14] .
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