RJ-MCMC with different options & tankwise metrics to compare approaches

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Detection and estimation of muons in the Auger project

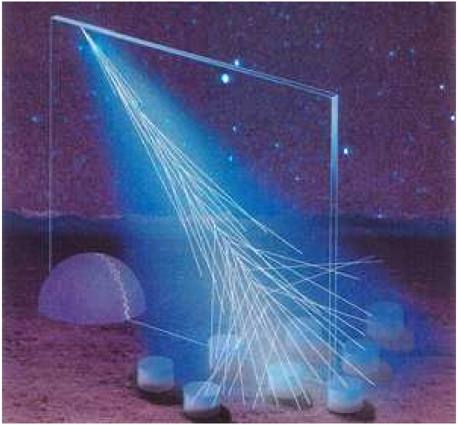
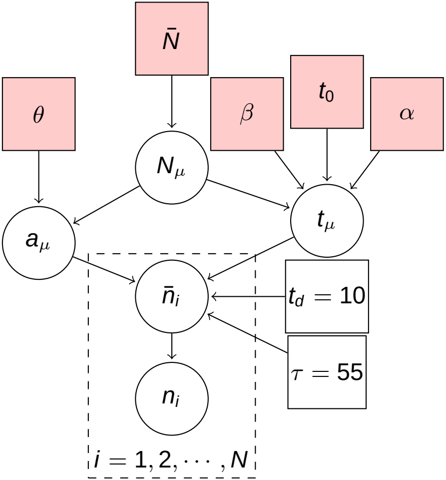


Figure: A conceptual shower (http://auger.org).

Tankwise muonic signal model



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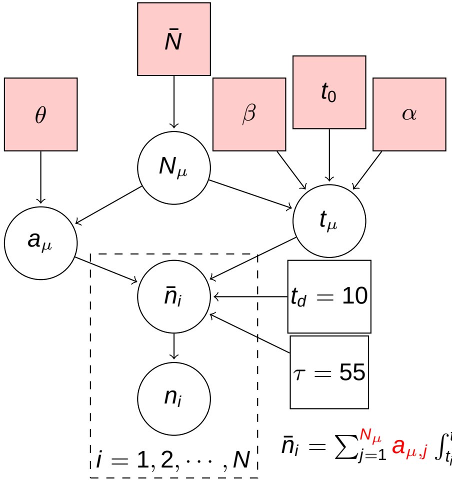

$$\bar{n}_i = \sum_{j=1}^{N_\mu} a_{\mu,j} \int_{t_{i-1}}^{t_i} p_{\tau,t_d}(t - t_{\mu,j})$$

Trans-dimensional problems

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- Space $\mathbb{X} = \bigcup_{N_\mu \in \mathrm{N}} \{N_\mu\} \times \boldsymbol{\Theta}^{N_\mu}$ with points $\boldsymbol{x} = (N_\mu, \boldsymbol{\theta}_{N_\mu})$

Outline

1. RJ-MCMC sampler
   - Within-model moves
   - Between models moves

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- Likelihood

$$p(\mathbf{x} \mid \mathbf{y}) = \frac{p(\mathbf{y} \mid \mathbf{x})\, p(\mathbf{x})}{\int_{\mathbb{X}} p(\mathbf{y} \mid \mathbf{x}')\, p(\mathbf{x}')\mathrm{d}\mathbf{x}'}$$

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- high-dimensional / intractable integrals

Markov Chain Monte Carlo (MCMC) methods

- generate samples from the posterior distribution of interest (target distribution) $p(\boldsymbol{x} \mid \mathbf{y})$.
- construct a Markov chain $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(M)})$ that under some conditions converges to $p(\boldsymbol{x} \mid \mathbf{y})$.

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- Famous algorithms:
  - Metropolis-Hastings (MH) sampler [Metropolis, et al. 1953, Hastings, 1970.]
  - Gibbs sampler [Geman and Geman, 1984.]
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Generalization of the famous Metropolis-Hastings (MH) sampler;

Given $\boldsymbol{x}^{(n)}$,

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i) Propose $\boldsymbol{x}' \sim Q(\boldsymbol{x}^{(n)}, \cdot)$.

ii) Accept, i.e., set $\boldsymbol{x}^{(n+1)} = \boldsymbol{x}'$, with probability $\alpha\left(\boldsymbol{x}^{(n)}, \boldsymbol{x}'\right)$;
Otherwise, reject the move $\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)}$.

RJ-MCMC sampler (Contd.)

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- Update the component-specific parameters, i.e., $\boldsymbol{a}_\mu$ and $\boldsymbol{t}_\mu$, assuming N_μ is fixed

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ii) Between models (trans-dimensional) moves
   - Propose jumps in the number N_μ of muons
   - Birth or Death moves: add or remove one muon

RJ-MCMC sampler: within-model moves

Given the number N_μ of muons is fixed

- propose new vectors of arrival times and amplitudes

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- propose new vectors of arrival times and amplitudes
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  - Normal random walk sampler, mixture of proposals, adaptive proposals (e.g., AMOR), ...
- Acceptance probability

$$\alpha(\boldsymbol{x}, \boldsymbol{x}') = \min \left\{ 1, \frac{p(\boldsymbol{x}' \mid \mathbf{y})}{p(\boldsymbol{x} \mid \mathbf{y})} \times \frac{Q(\boldsymbol{x} \mid \boldsymbol{x}')}{Q(\boldsymbol{x}' \mid \boldsymbol{x})} \right\}$$

Normal Random Walk sampler:

- $Q(\cdot \mid \boldsymbol{x}) = \mathcal{N}(\cdot \mid \boldsymbol{x}, \sigma^2)$
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- the performance depends on the scale parameter σ^2

First example

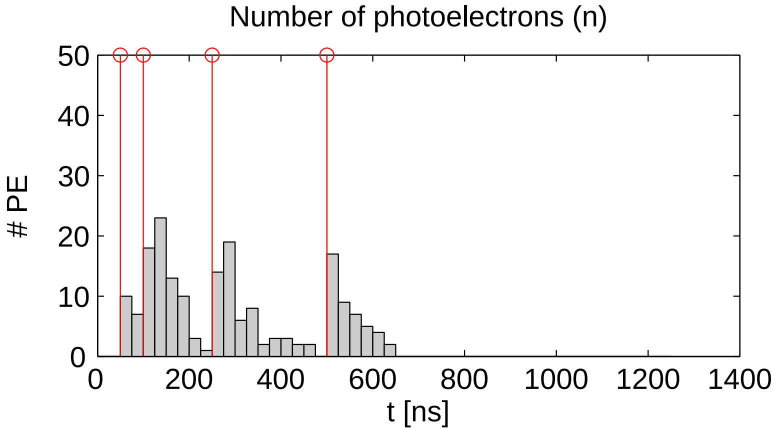
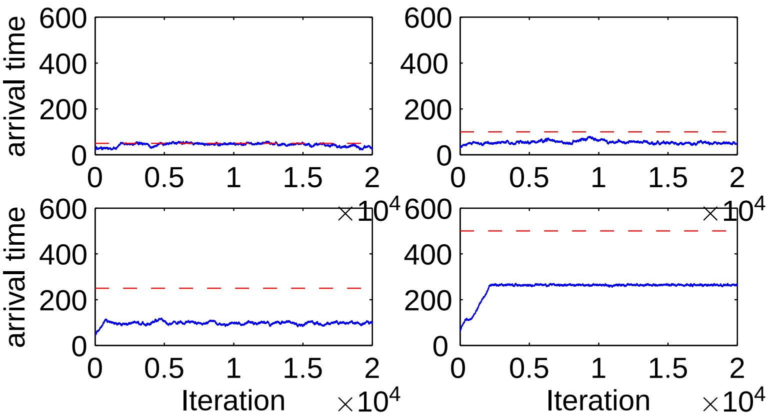
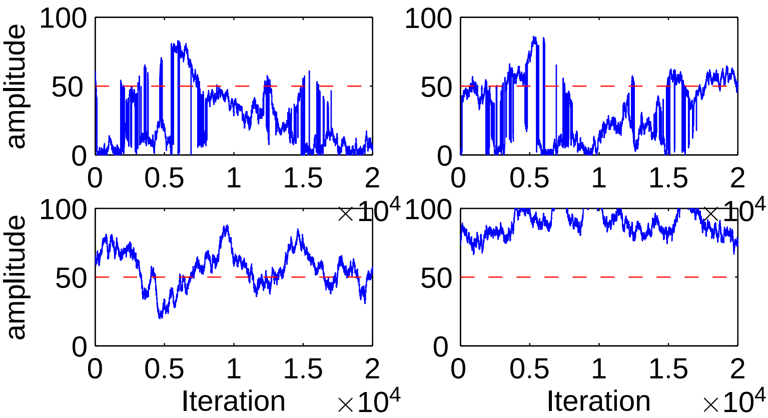


Figure: Observed signal n when $N_\mu = 4$, $\boldsymbol{t}_\mu = (50, 100, 250, 500)^t$ and $\boldsymbol{a}_\mu = (50, 50, 50, 50)^t$.

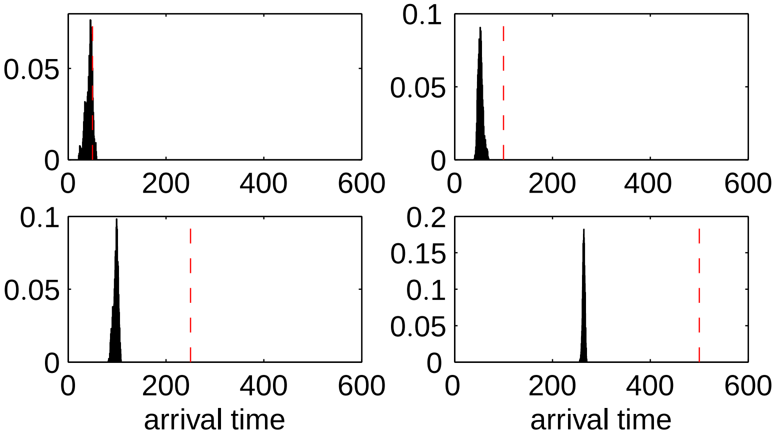
Trace of sorted arrival times (small $\sigma = 5$)



Trace of (sorted) amplitudes (small $\sigma = 5$)



Histogram of sorted arrival times (small $\sigma = 5$)



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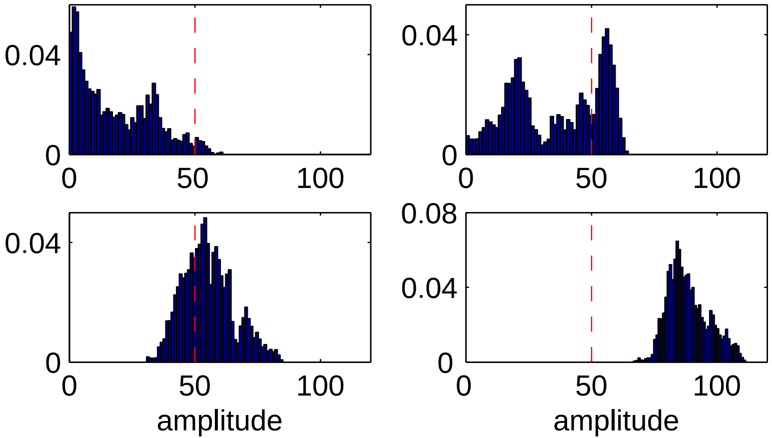
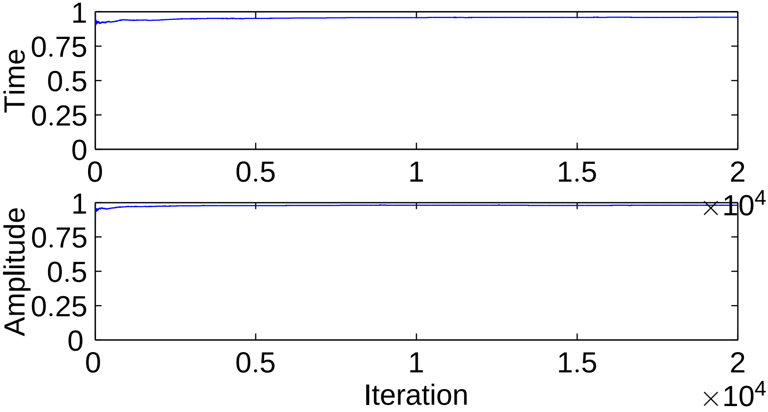
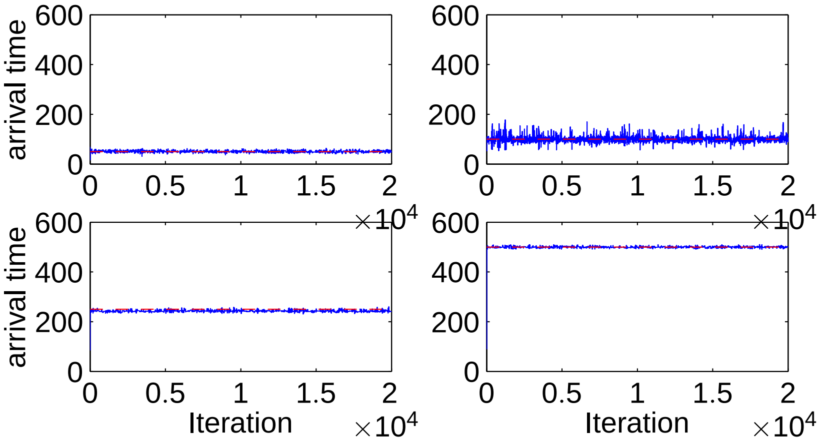


Average MH acceptance rates (small $\sigma = 5$)



Trace of sorted arrival times (large $\sigma = 100$)



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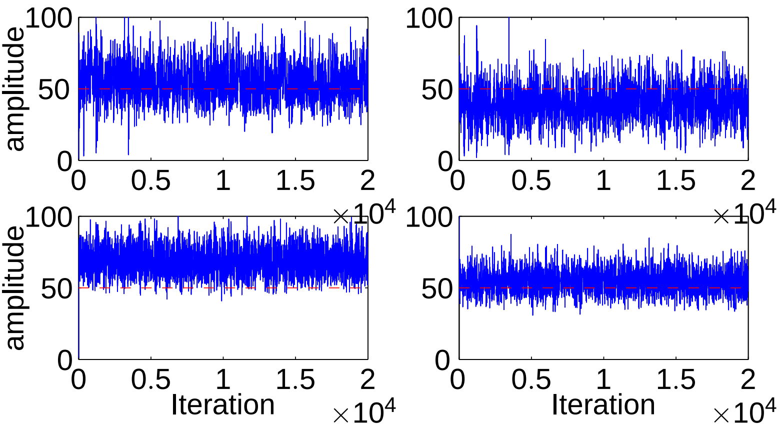

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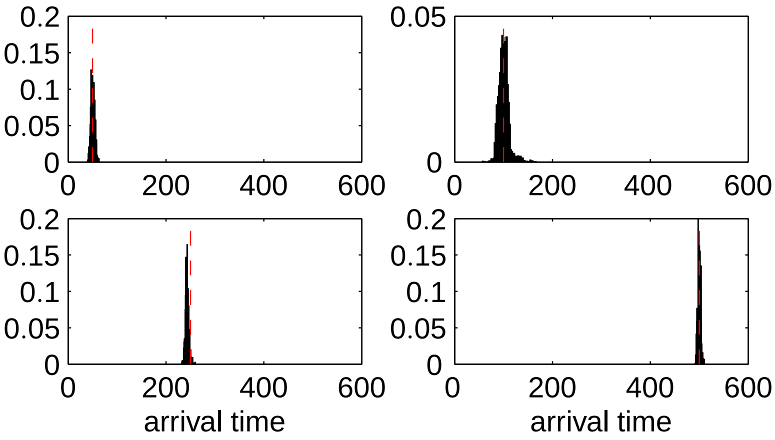

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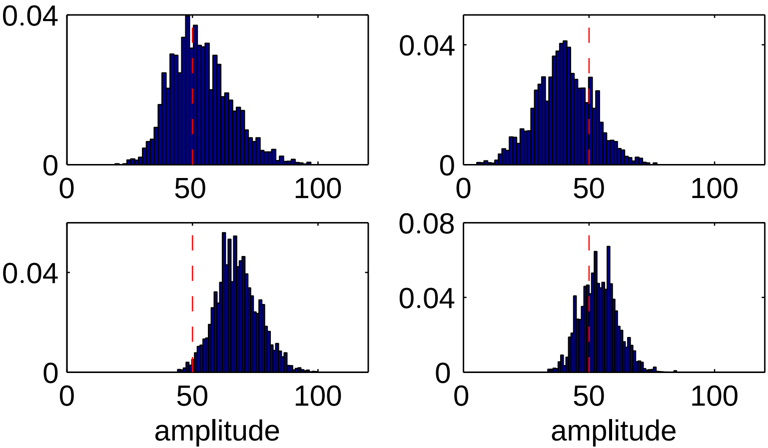

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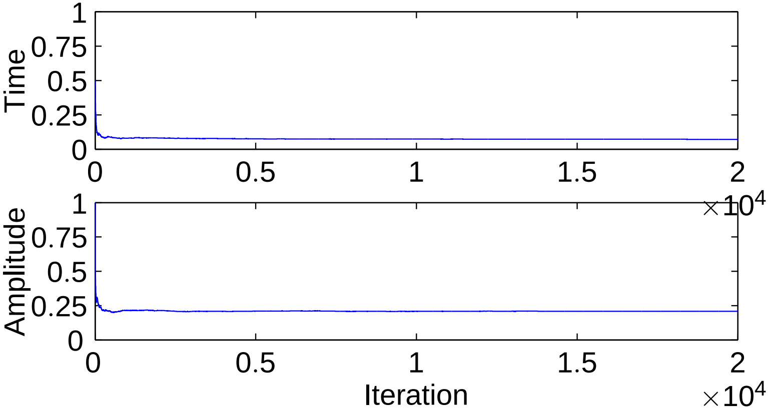

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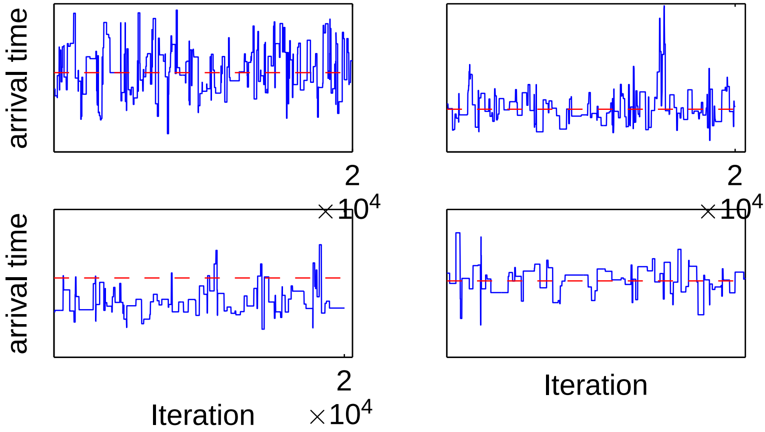

Some “simple” solutions

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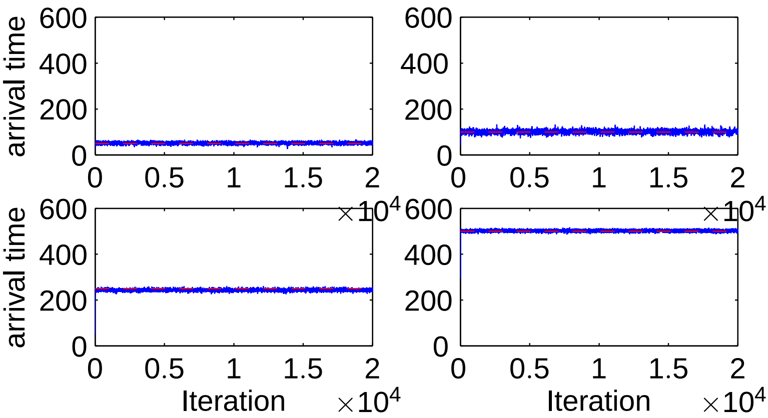
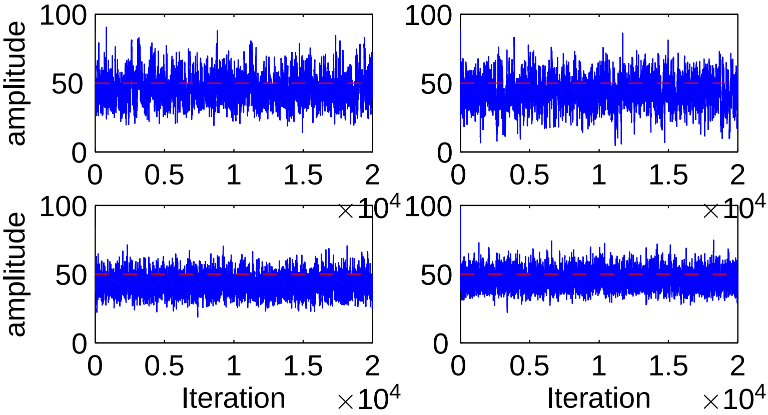
  - $\alpha(\boldsymbol{x}, \boldsymbol{x}') = \min\left\{1, \frac{p(\boldsymbol{x}'|\mathbf{y})}{p(\boldsymbol{x}|\mathbf{y})}\right\}$
- Local move $\Rightarrow \sigma_1 = 5$
- Global move $\Rightarrow \sigma_2 = 100$
- $\lambda = 0.8$

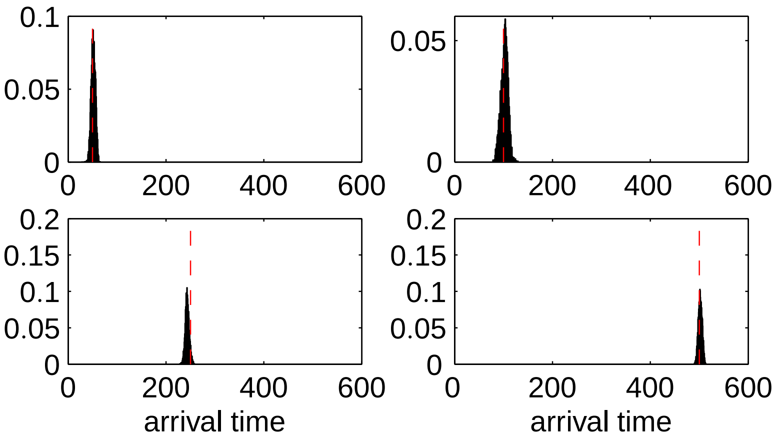
Trace of sorted arrival times (Mixture)



Trace of (sorted) amplitudes (Mixture)



Histogram of sorted arrival times (Mixture)



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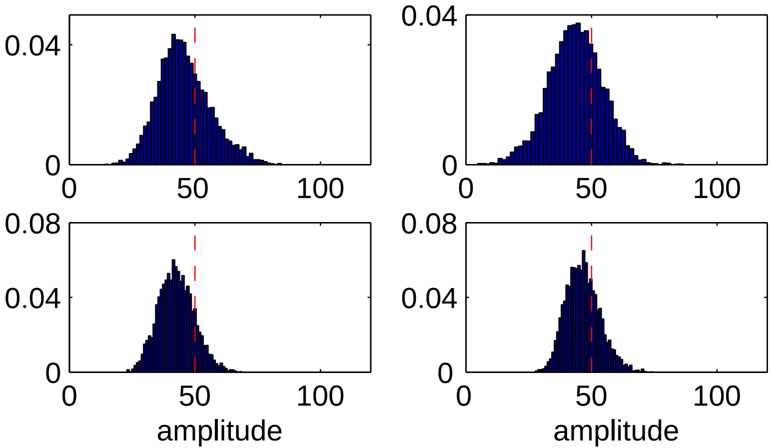
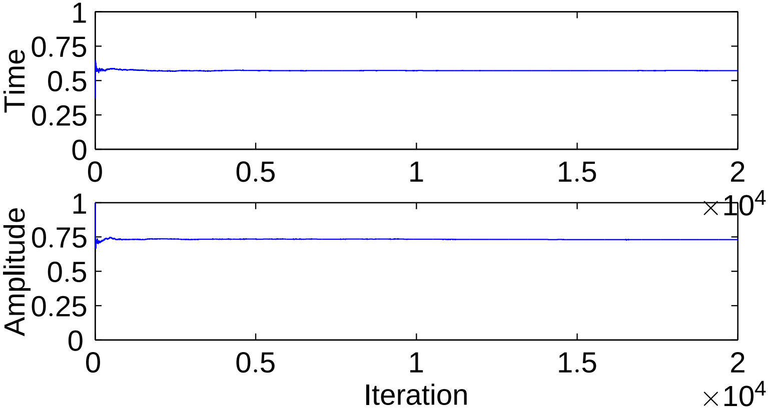


Average MH acceptance rates (Mixture)



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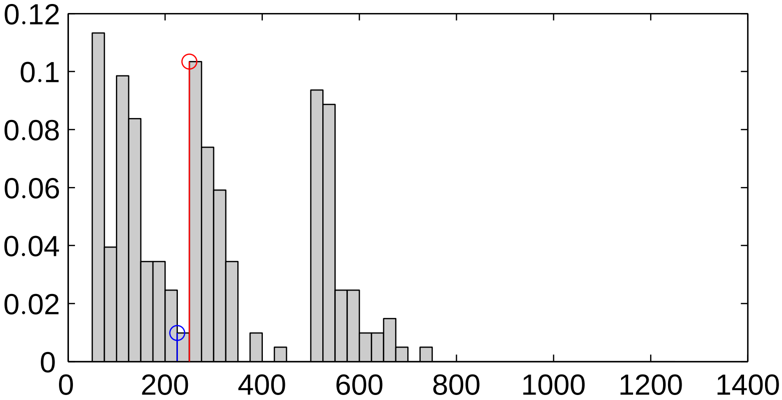
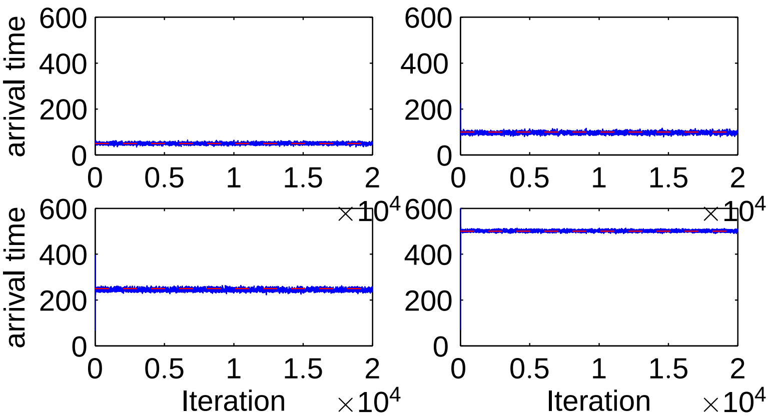


Figure: Normalized observed data (n). Blue is the current state and red is the proposed state.

Trace of sorted arrival times (Obs. Data)



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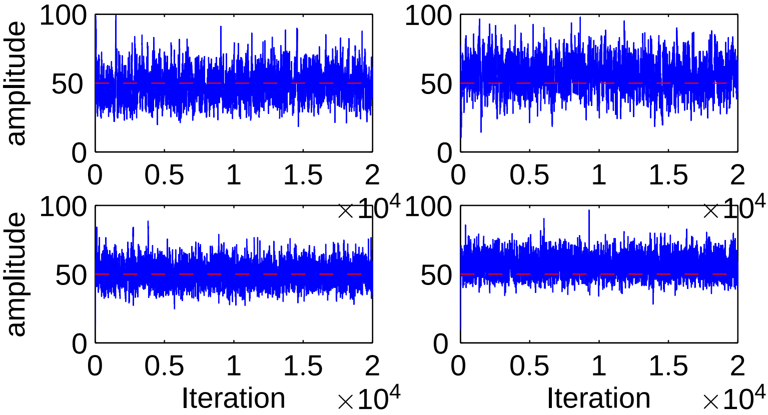

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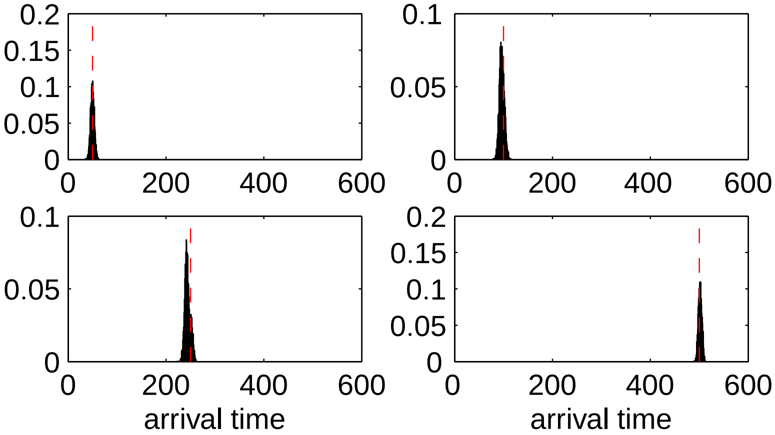

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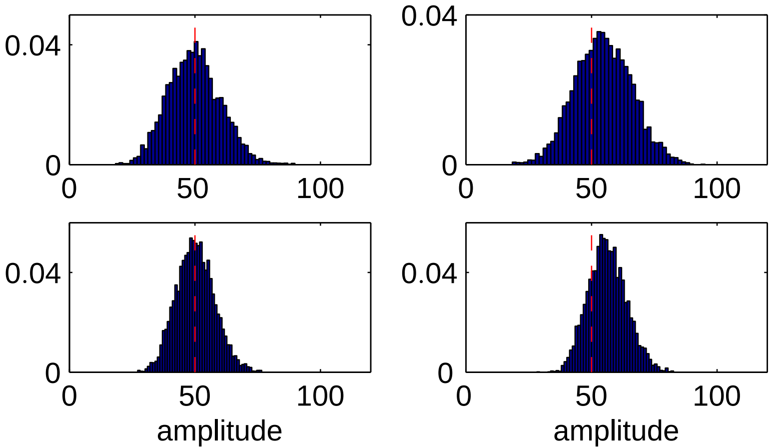
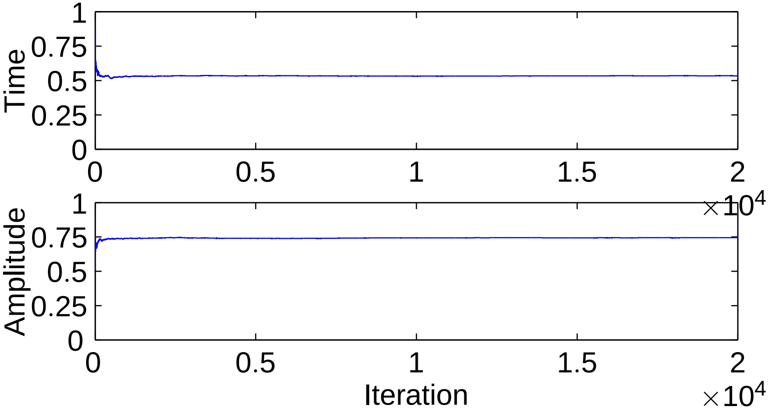


Average MH acceptance rates (Obs. Data)



Between models moves

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- Death move: removes a muon ($\mathcal{M}_{N_\mu} \to \mathcal{M}_{N_\mu-1}$)

Between models moves (Contd.)

- The performance of the RJ-MCMC sampler depends on $Q(\theta^\star) = Q(a_\mu^\star) \cdot Q(t_\mu^\star)$

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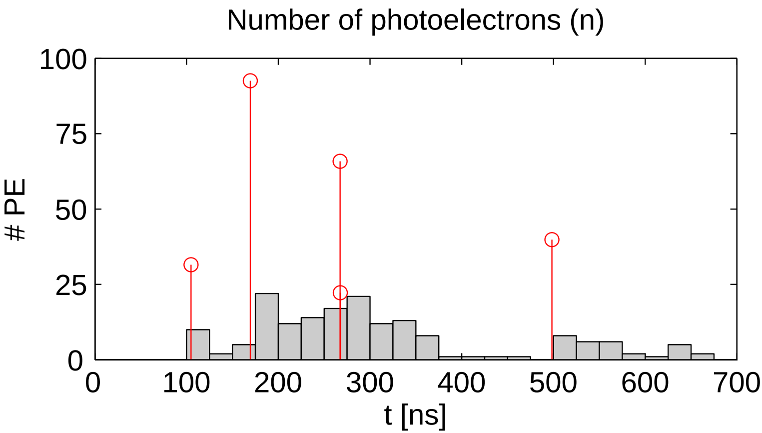
  - Using residual of the reconstructed signal
  - Adaptive RJ-MCMC sampler

Example 1: Observed data (n)



Example 1: AR RJ-MCMC sampler

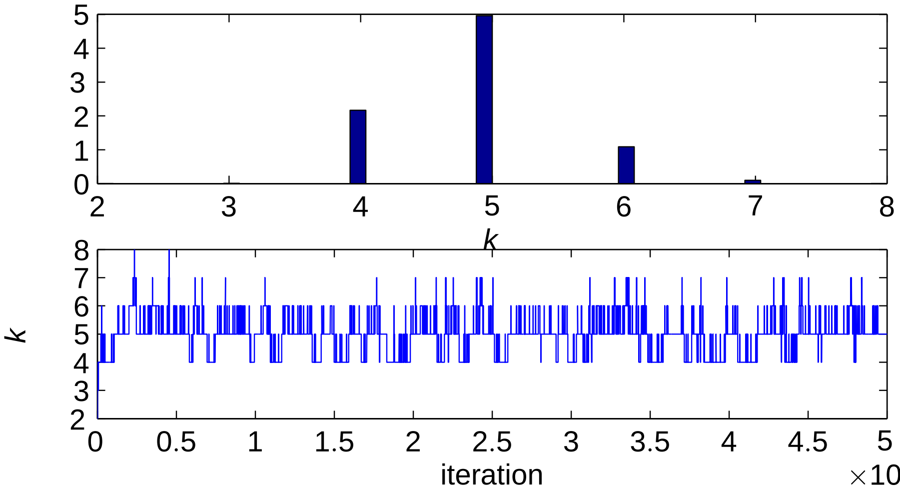


Figure: Posterior and chain of N_μ.

Example 1: AR RJ-MCMC sampler (Contd.)

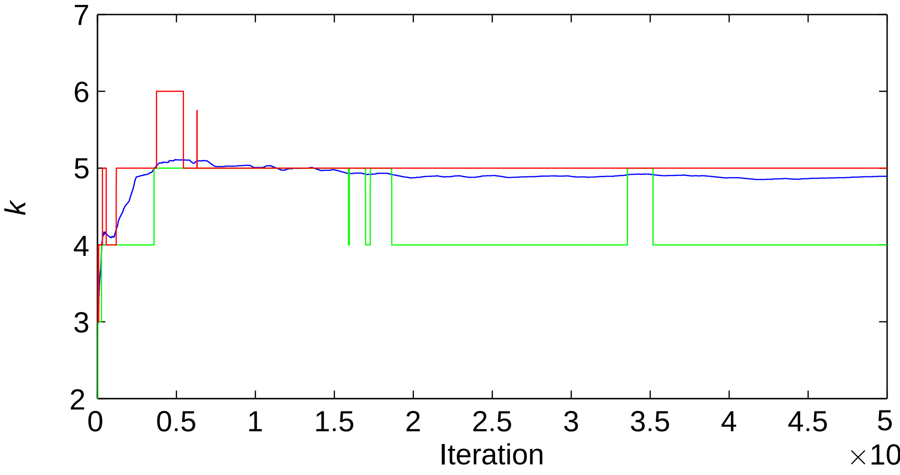


Figure: Statistics of the chain of N_μ; mean and 25 confidence

Example 1: AR RJ-MCMC sampler (Contd.)

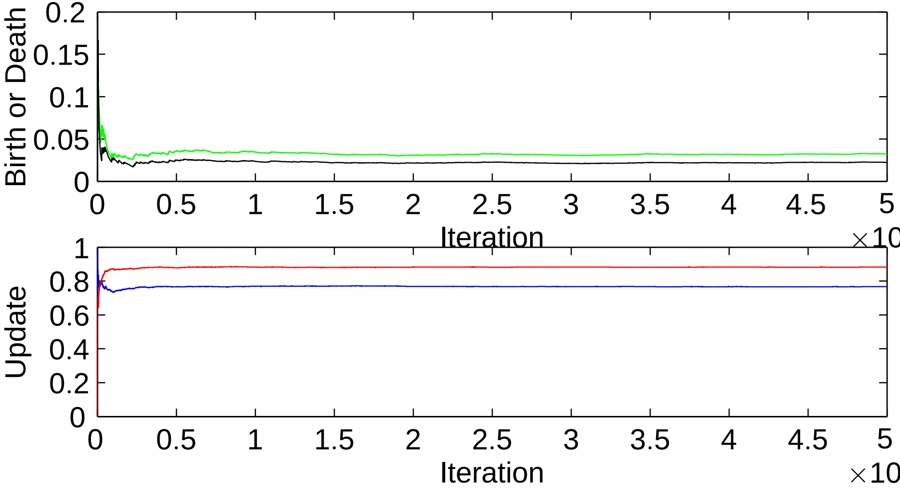


Figure: Average MH acceptance probabilities.

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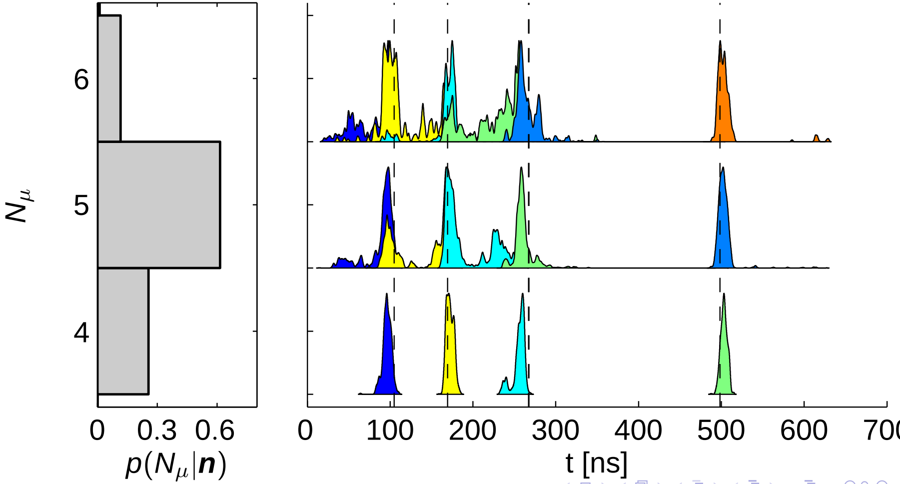
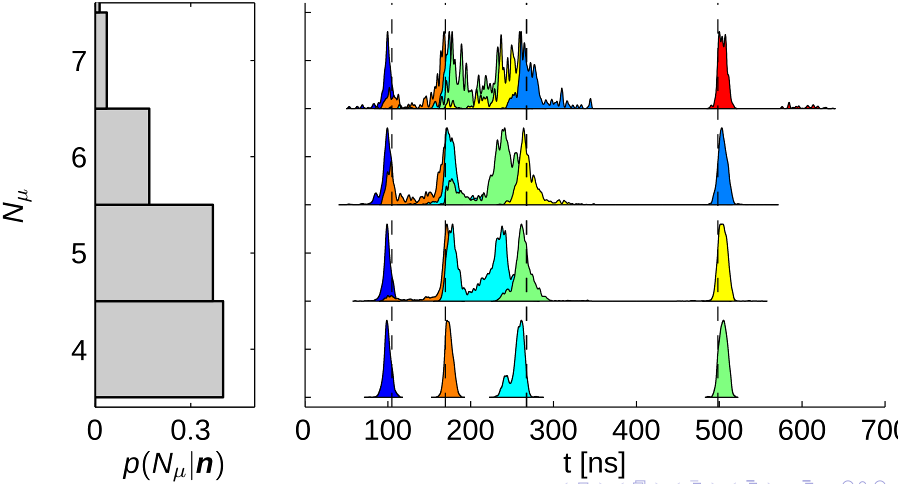


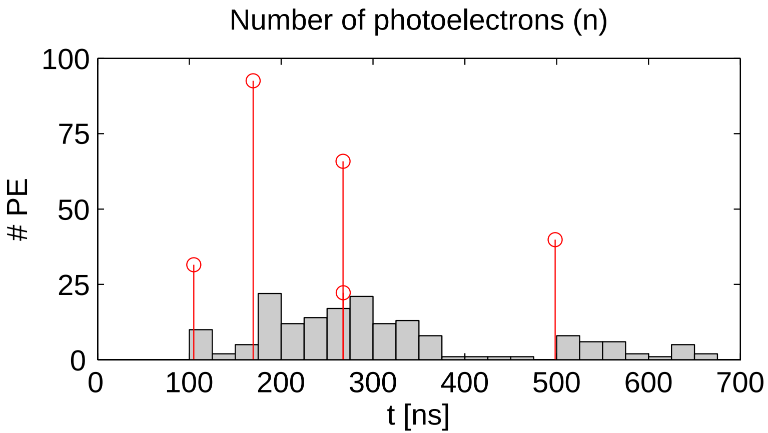
Example 1: BK & RB RJ-MCMC sampler



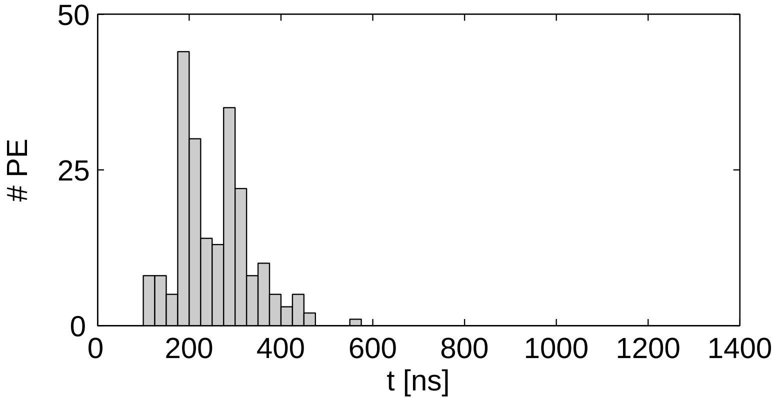
Between models moves (Contd.)

- The performance of the RJ-MCMC sampler depends on $Q(\theta^\star) = Q(a^\star_\mu) \cdot Q(t^\star_\mu)$
  - Using their prior distributions, i.e., $Q(a^\star_\mu) = \mathcal{U}(0, 134.4)$ and $Q(t^\star_\mu) = \mathcal{IG}(2.5, 350)$
  - Using residual of the reconstructed signal
  - Adaptive RJ-MCMC sampler

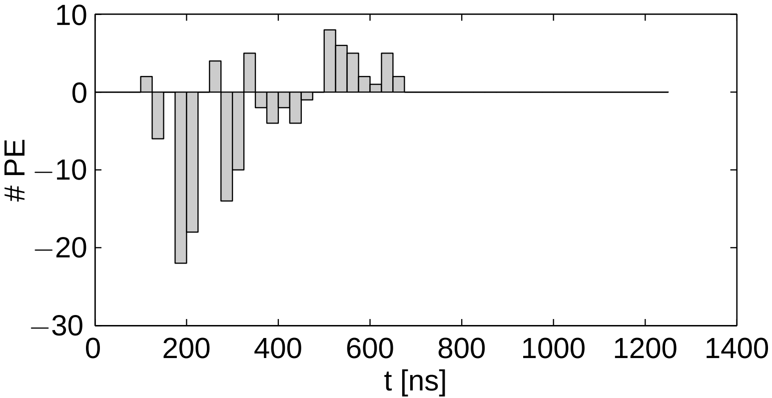
Example 1: Observed data (n)



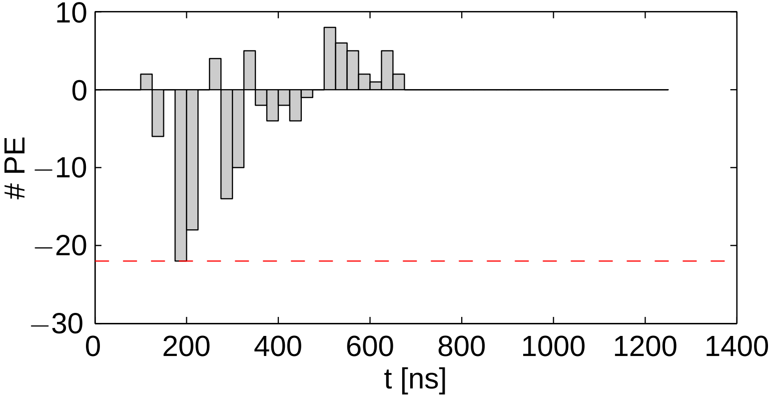
Current reconstructed signal ($\hat{n}$)



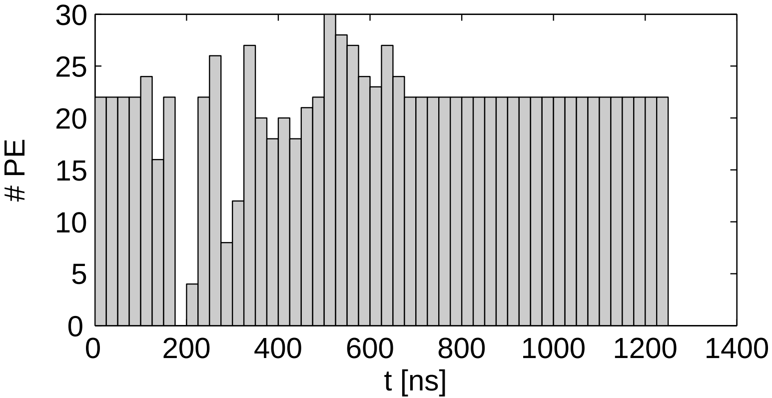
Residual $(n - \hat{n})$



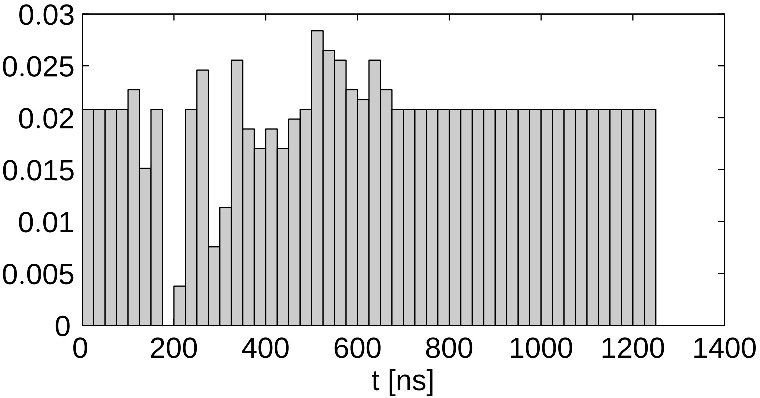
Residual $(n - \hat{n})$



Biased residual



PDF to sample from!



Outline

Metrics

- Which method performs better than the others?

Metrics

- Which method performs better than the others?
- Distance measures or metrics?

Metrics: Fixed-dimensional case

- Many distance measures:

Metrics: Fixed-dimensional case

- Many distance measures:
  - mean squared error

$$MSE = \sqrt{(\theta - \hat{\theta})^t (\theta - \hat{\theta})}$$

Metrics: Fixed-dimensional case

- Many distance measures:
  - ➠ mean squared error

$$MSE = \sqrt{(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^t (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})}$$

Setting $\boldsymbol{\theta} = (\boldsymbol{t}_\mu, \boldsymbol{a}_\mu)$, then

	$\sigma = 5$	$\sigma = 100$	Mixture	Observe	AMOR
MSE	289.55	22.01	14.11	**10.81**	?

But what can we conclude?

Metrics: Trans-dimensional case

Challenges:

- Simulation based methods

Metrics: Trans-dimensional case

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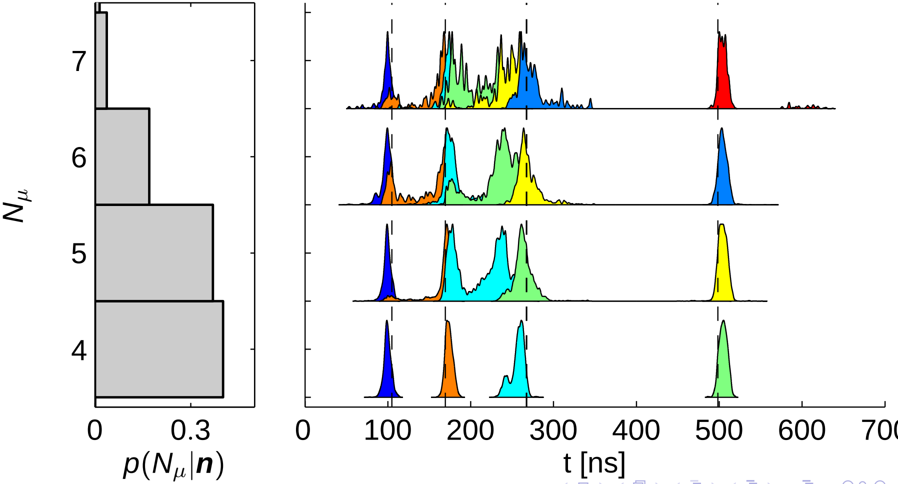
- Simulation based methods
  - ➠ how to report point estimates (summarize posterior)?

Metrics: Trans-dimensional case

Challenges:

- Simulation based methods
  - ➠ how to report point estimates (summarize posterior)?
  - ➠ label-switching issue

Example 1: BK & RB RJ-MCMC sampler



Summarized posterior

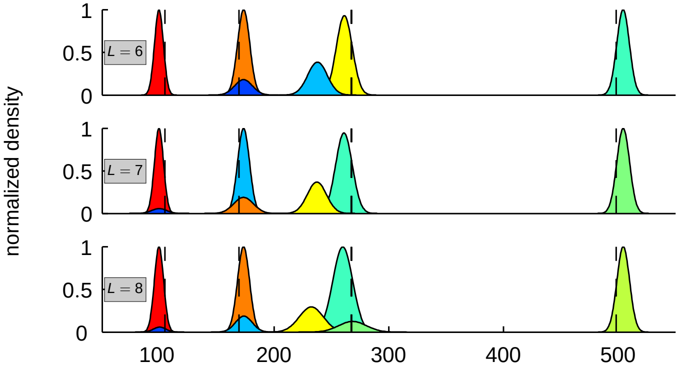


Figure: Normalized densities of the fitted Gaussian components.

Metrics: Trans-dimensional case

Challenges:

- Simulation based methods
  - ➠ How to report point estimates (summarize posterior)?
  - ➠ label-switching issue
- How to compare vectors of different dimensions?

Metrics: Trans-dimensional case (Contd.)

Some ideas

- Work with quantities that depend neither on the labels nor on the dimensions

Metrics: Trans-dimensional case (Contd.)

Some ideas

- Work with quantities that depend neither on the labels nor on the dimensions
  - ➠ reconstructed signal (not very good due to randomness)

Metrics: Trans-dimensional case (Contd.)

Some ideas

- Work with quantities that depend neither on the labels nor on the dimensions
  - ➠ reconstructed signal (not very good due to randomness)
  - ➠ estimating hyperparameters, e.g., hyperparameters of prior over arrival times

Metrics: Trans-dimensional case (Contd.)

Some ideas

- Work with quantities that depend neither on the labels nor on the dimensions
  - reconstructed signal (not very good due to randomness)
  - estimating hyperparameters, e.g., hyperparameters of prior over arrival times
  - optimization team?

Metrics: Trans-dimensional case (Contd.)

Some ideas

- binned parameter space

Metrics: Trans-dimensional case (Contd.)

Some ideas

- binned parameter space
  - ➠ counting False Alarms and Omissions to construct a Loss function

Metrics: Trans-dimensional case (Contd.)

Some ideas

- binned parameter space
  - ➠ counting False Alarms and Omissions to construct a Loss function
- Other distance measures

Outline

1. RJ-MCMC sampler
   - Within-model moves
   - Between models moves
2. Metrics
   - Fixed-dimensional case
   - Trans-dimensional case
3. Conclusion

Conclusion & Future work

- The mixture and observed data driven proposals had "good" performance

Conclusion & Future work

- The mixture and observed data driven proposals had "good" performance
- The between model moves' acceptance rates were low although the RJ-MCMC sampler mixed well?!

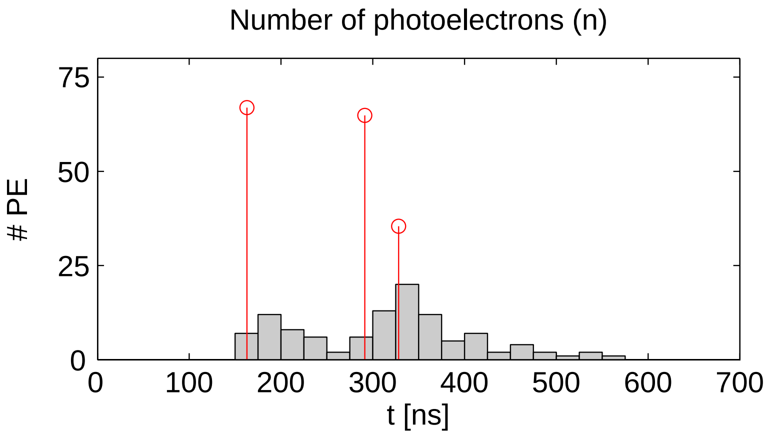
Conclusion & Future work

- improving the between model moves proposals
  - Observed data driven proposals
  - Adaptive RJ-MCMC sampler?
- Sampler for the complete model
  - Electromagnetic components
  - Real observed signal
- implementing metrics to compare methods

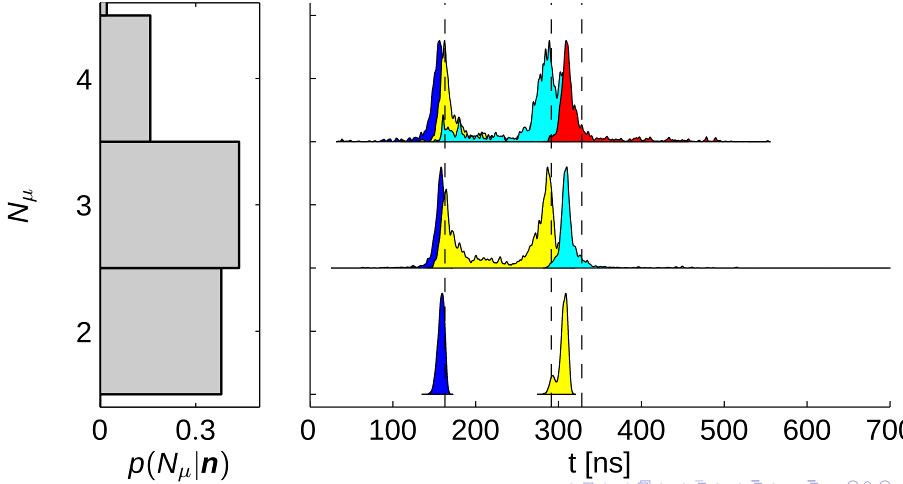
Questions?

Thank you for your attention!

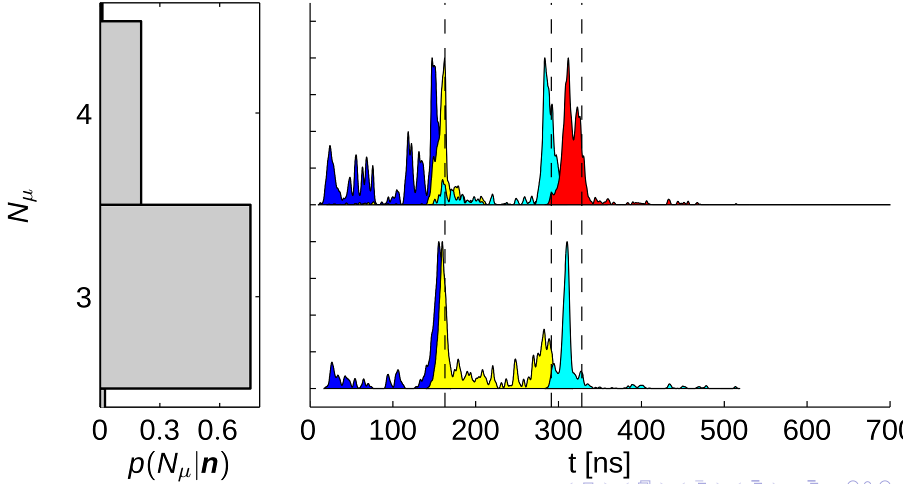
Example 2: Observed data (*n*)



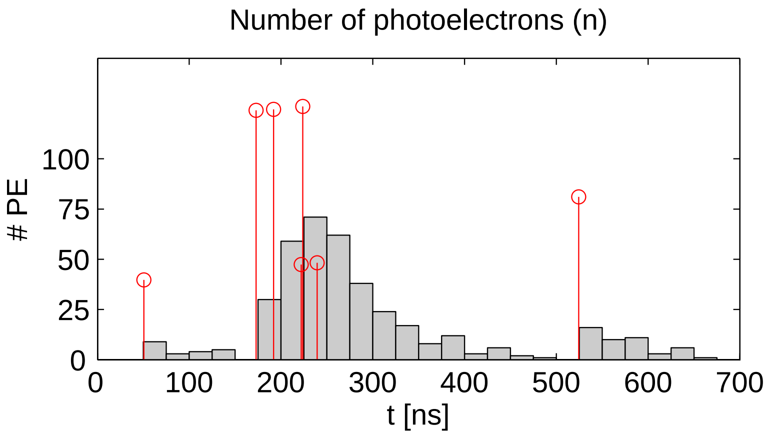
Example 2: BK & RB RJ-MCMC sampler



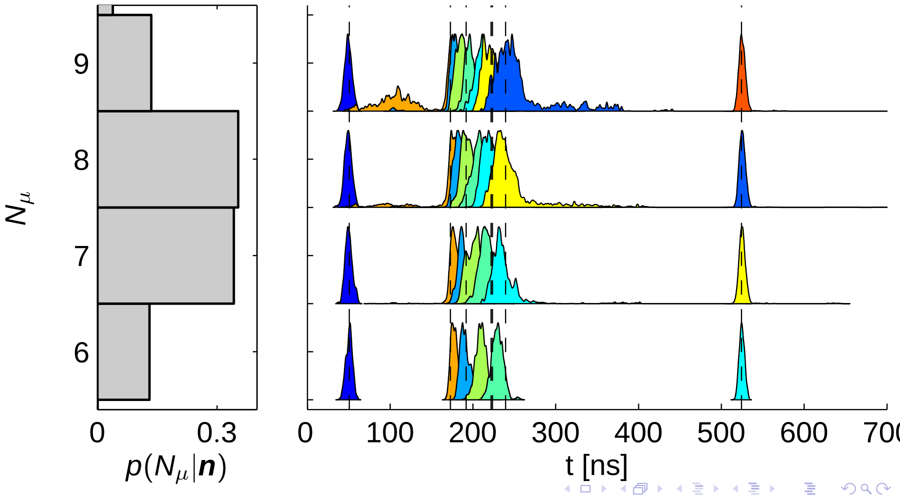
Example 2: AR RJ-MCMC sampler



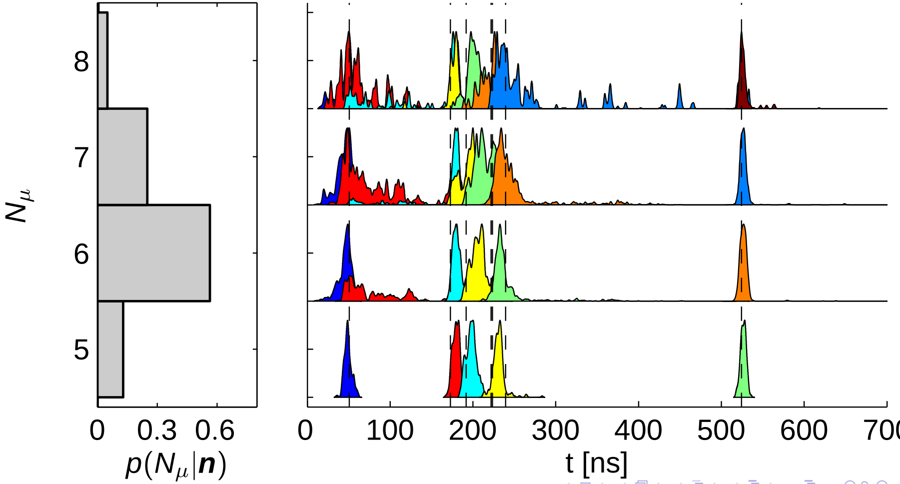
Example 3: Observed data (n)



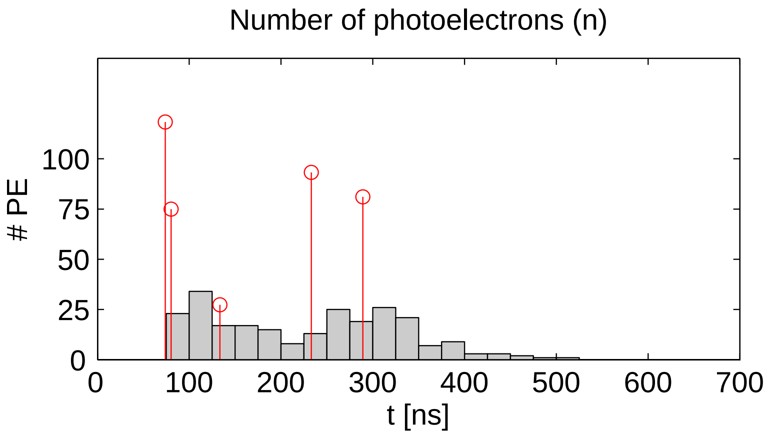
Example 3: BK & RB RJ-MCMC sampler



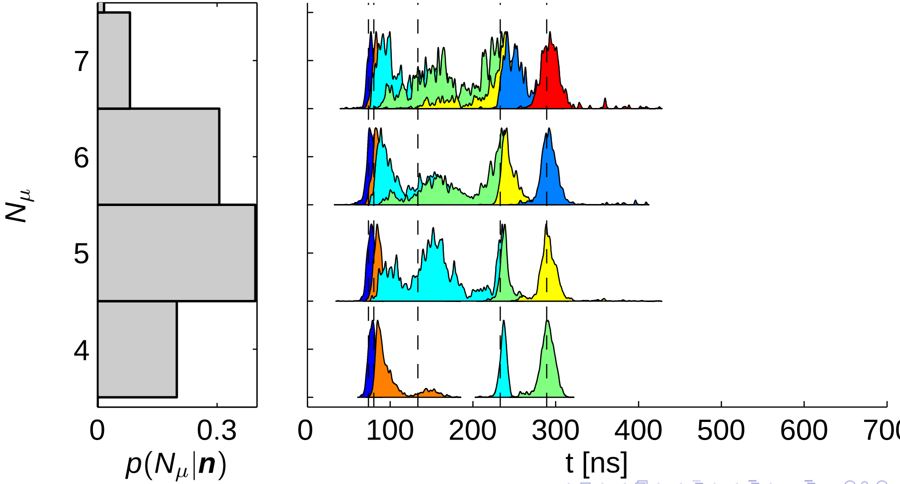
Example 3: AR RJ-MCMC sampler



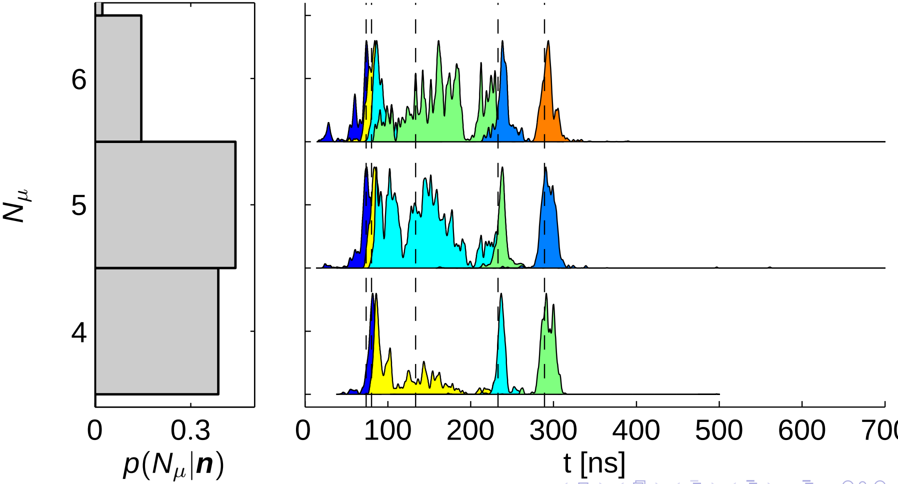
Example 4: Observed data (n)



Example 4: BK & RB RJ-MCMC sampler



Example 4: AR RJ-MCMC sampler



Example 6: Observed data (n)

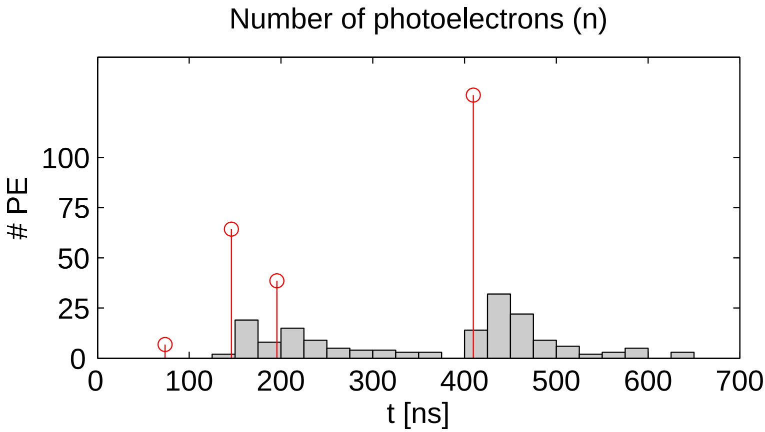
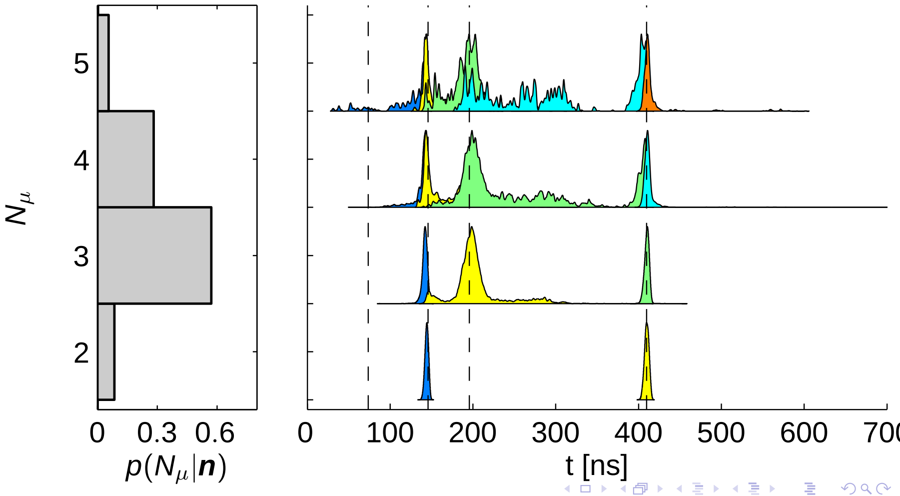
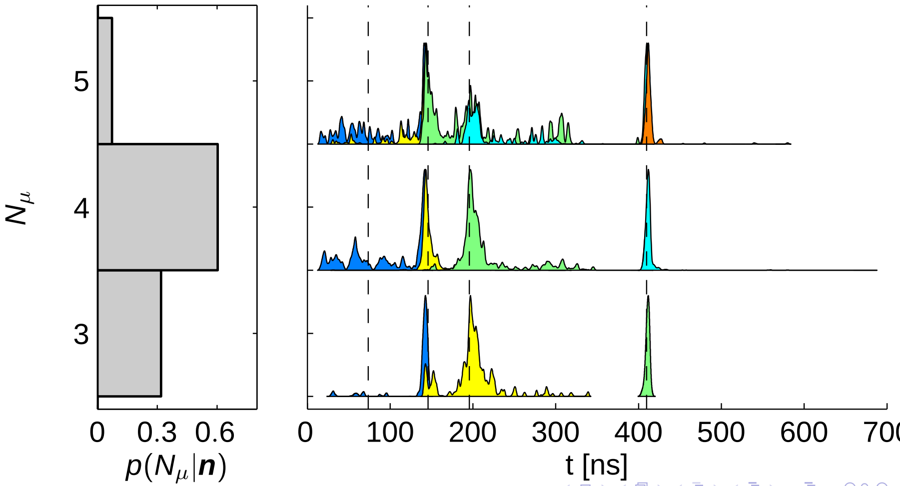


Figure: Posteriors of N_μ and sorted arrival times, $\boldsymbol{t}_\mu$, given N_μ.

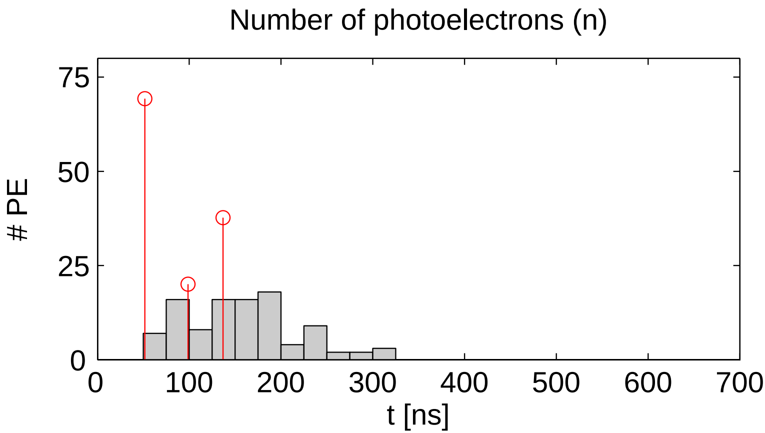
Example 6: BK & RB RJ-MCMC sampler



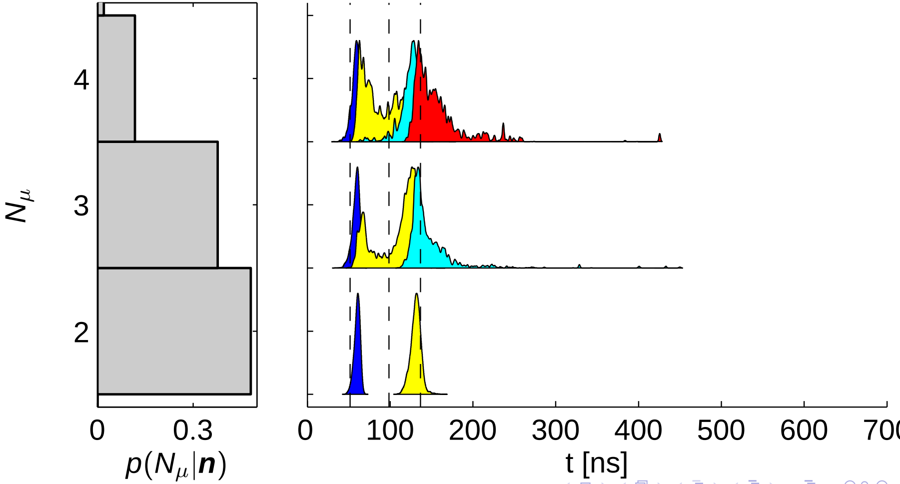
Example 6: AR RJ-MCMC sampler



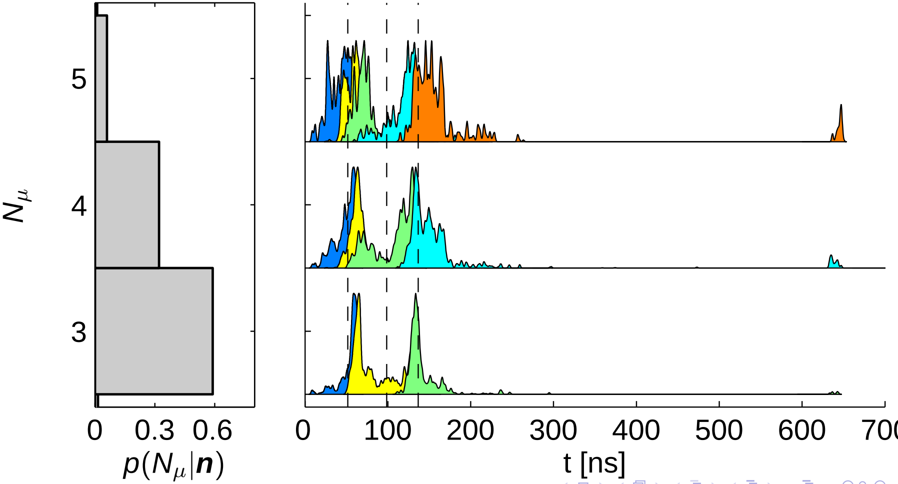
Example 7: Observed data (n)



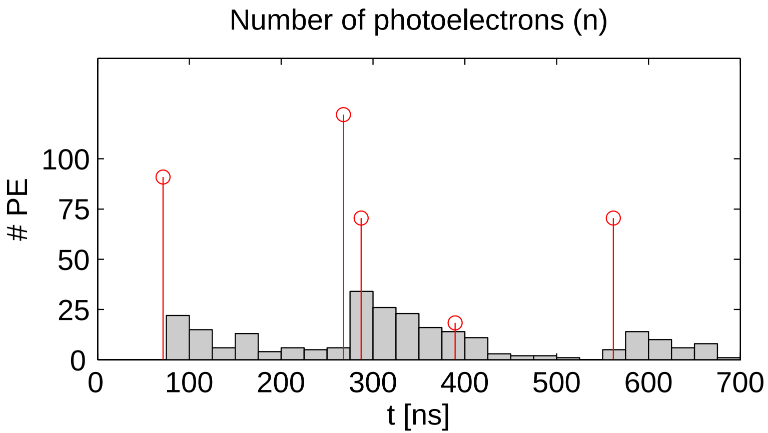
Example 7: BK & RB RJ-MCMC sampler



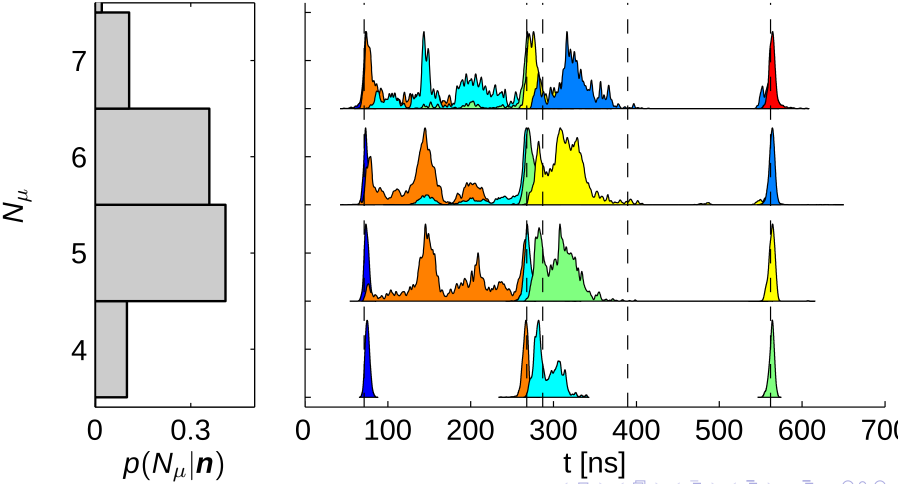
Example 7: AR RJ-MCMC sampler



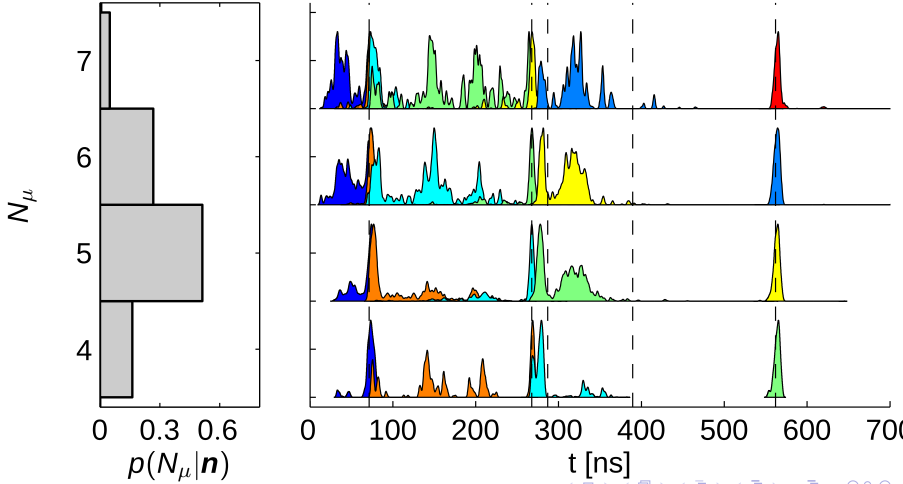
Example 8: Observed data (n)



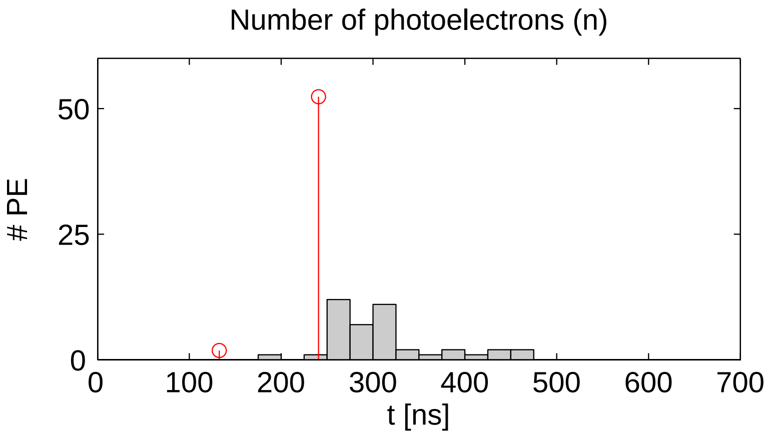
Example 8: BK & RB RJ-MCMC sampler



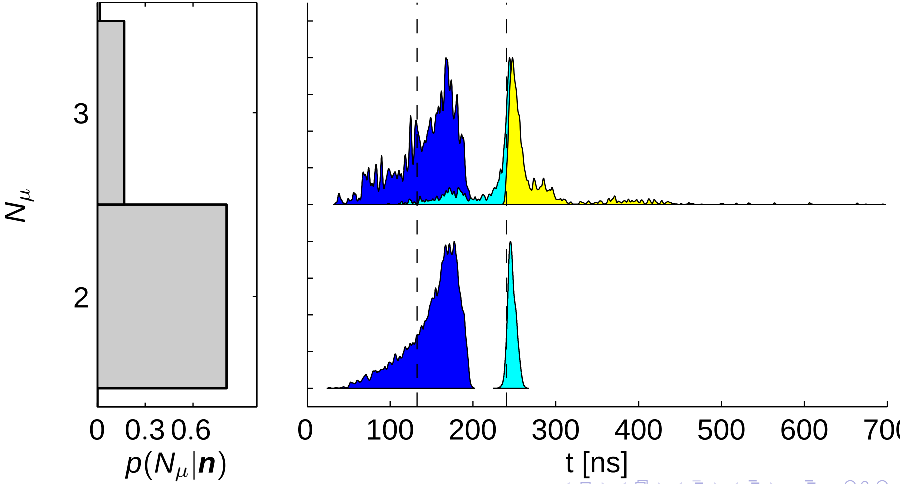
Example 8: AR RJ-MCMC sampler



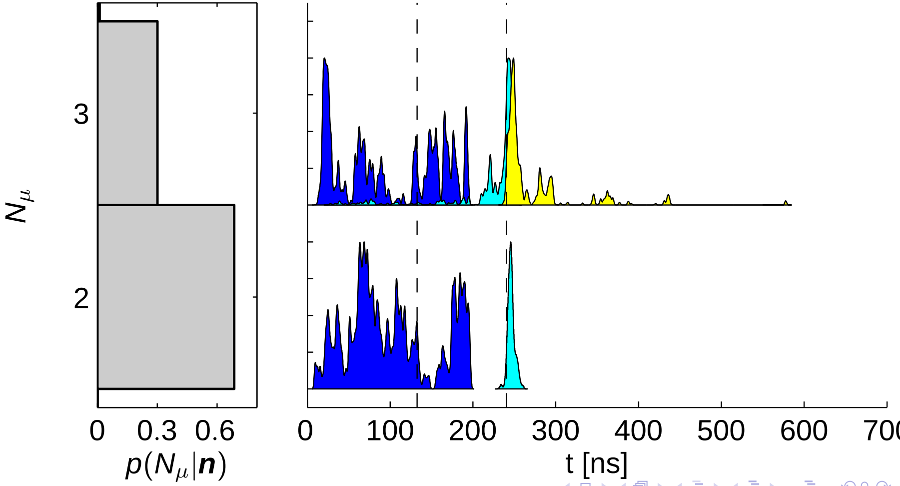
Example 9: Observed data (n)



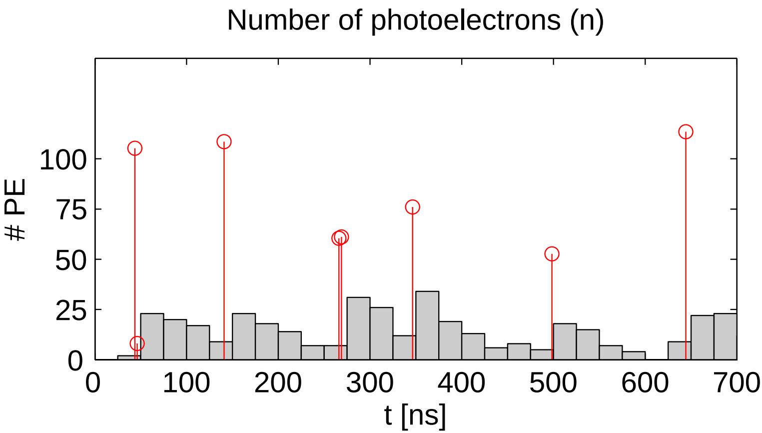
Example 9: BK & RB RJ-MCMC sampler



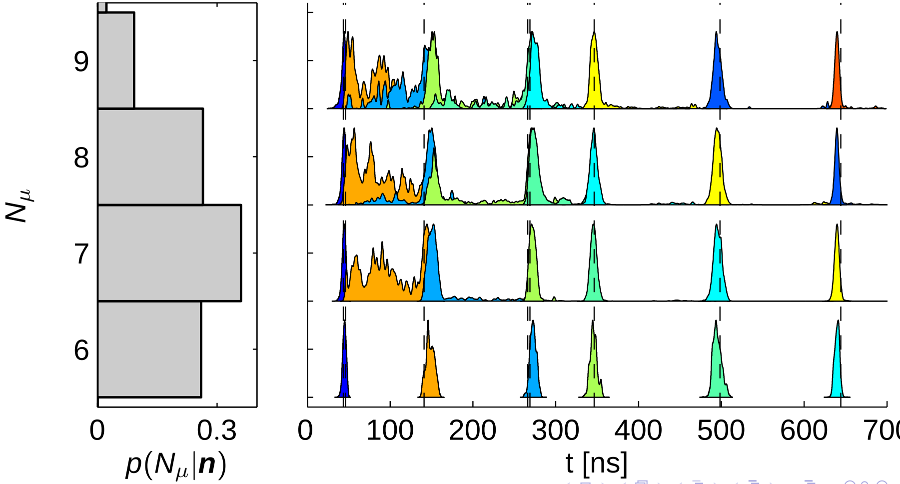
Example 9: AR RJ-MCMC sampler



Example 10: Observed data (*n*)



Example 10: BK & RB RJ-MCMC sampler



Example 10: AR RJ-MCMC sampler