Light and Shadow amongst QCD and QED
Montpellier — 16/17 November 2016

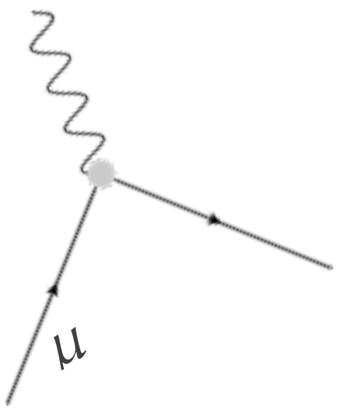
Lattice calculations of the leading hadronic contribution to the muon g-2

Andreas Jüttner

University of Southampton

Magnetic moment of the muon

photon interacting with a static magnetic field

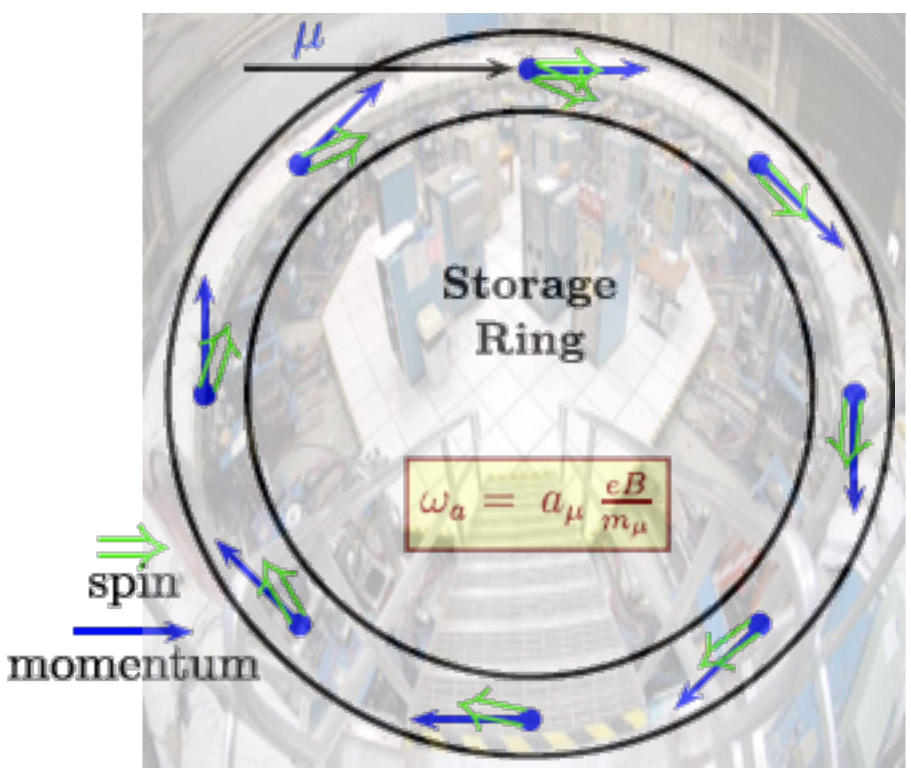


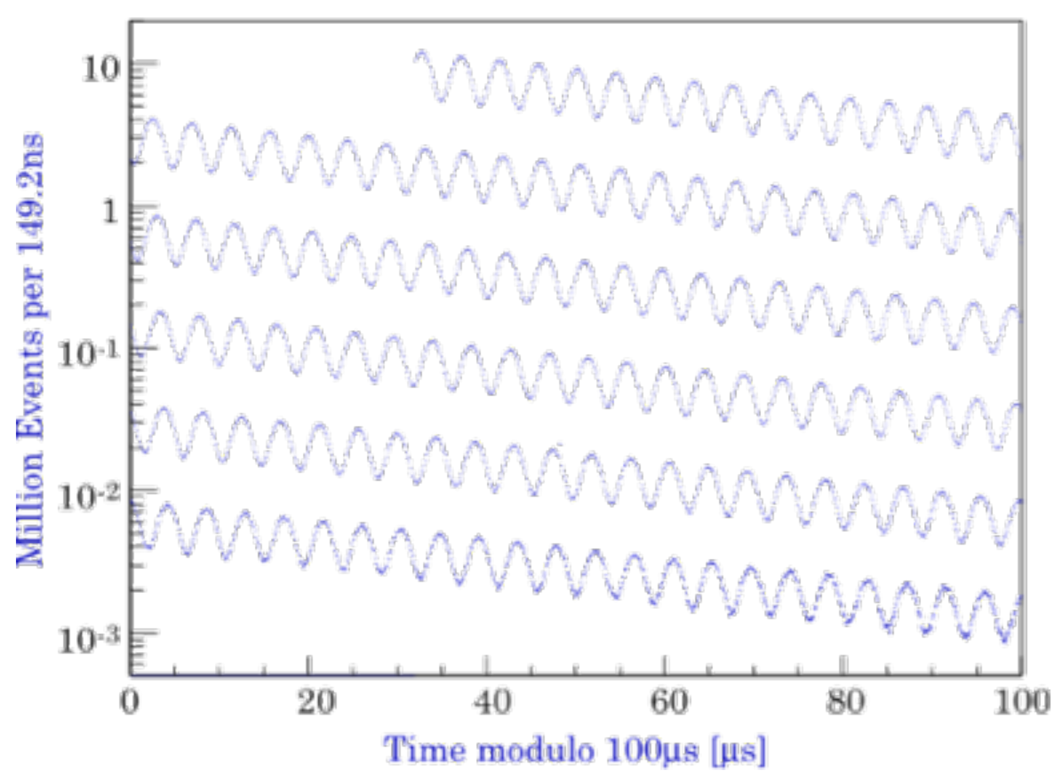
$$V = -\vec{\mu} \cdot \vec{B}$$

$$\vec{\mu} = g \left(\frac{e}{2m} \right) \vec{S}$$

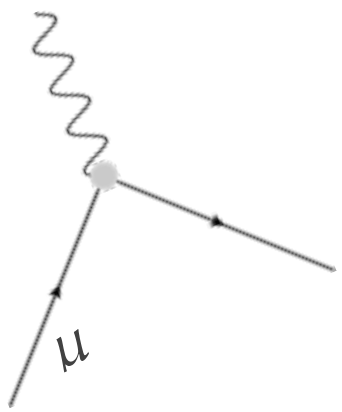
Landé g-factor
classically g=2

Experiment: BNL E821





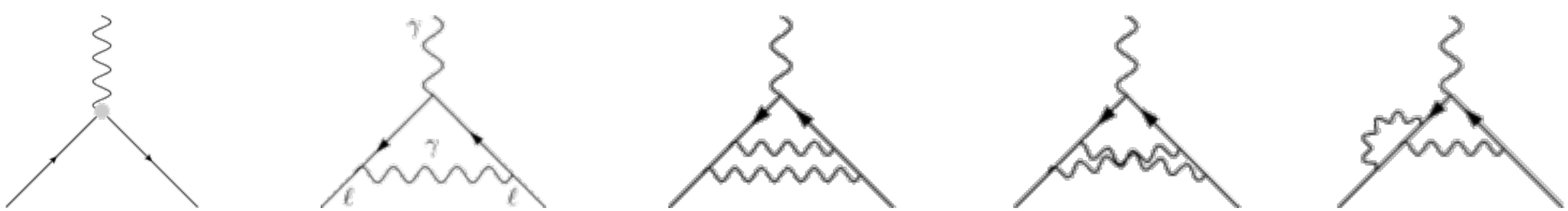
Magnetic moment of the muon



$$\langle l(\vec{p}\,')|j_\nu|l(\vec{p})\rangle = -e\,\bar{u}(\vec{p}\,')\left[F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu,\gamma_\rho]q_\rho\right]u(\vec{p})$$

$$F_2(0) = \frac{g-2}{2} \equiv a_\mu$$

Theory:



...

Magnetic moment of the muon

	central value x 10^{10}	uncertainty x 10^{10}
QED	11658471.895	0.008
EW	15.4	0.1
QCD LO	692.3	4.2
QCD NLO	-9.84	0.06
QCD NNLO	1.24	0.01
QCD LbL	10.5	2.6
SM TOTAL	11659181.5	4.9
Experiment	11659209.1	6.3

PDG

Tremendous success of Quantum Field Theory!!!

BSM contributions can be sizeable: $\delta a_\mu \propto m_\mu^2/M^2$

- new heavy states?
- extra dimensions?
- super symmetry?
- statistical fluctuation?

Magnetic moment of the muon

New experiments:

- Fermilab E989, early 2017, 0.14ppm
- J-PARC E34 later, aims for 0.3-0.4ppm, eventually 0.1ppm

SM TOTAL	11659181.5	4.9
Experiment	11659209.1	6.3

Fermilab **1.6**

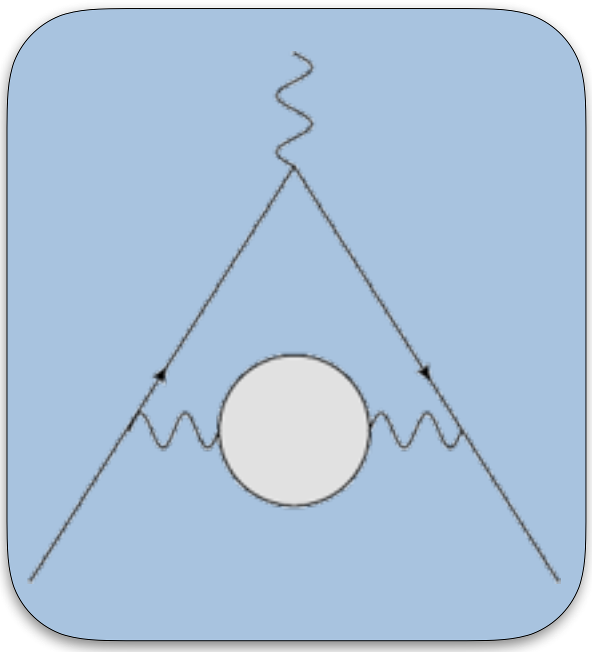
J-PARC **4.3 (later ~1)**

More precise theory prediction for hadronic contributions needed!

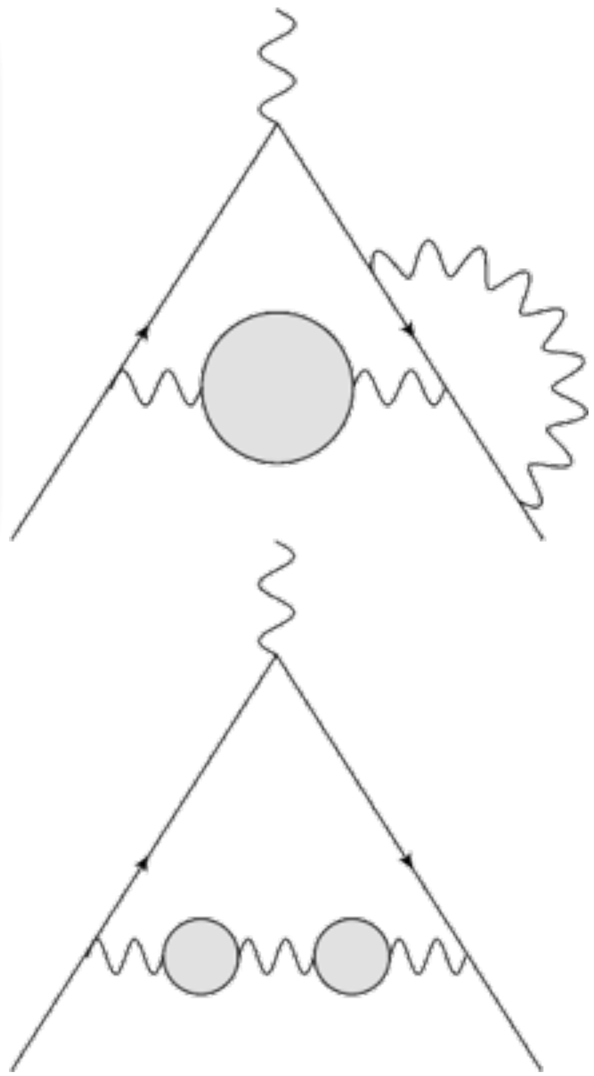
Aim at ~1%(10%) precision for QCD LO(LbL)

Hadronic Contributions

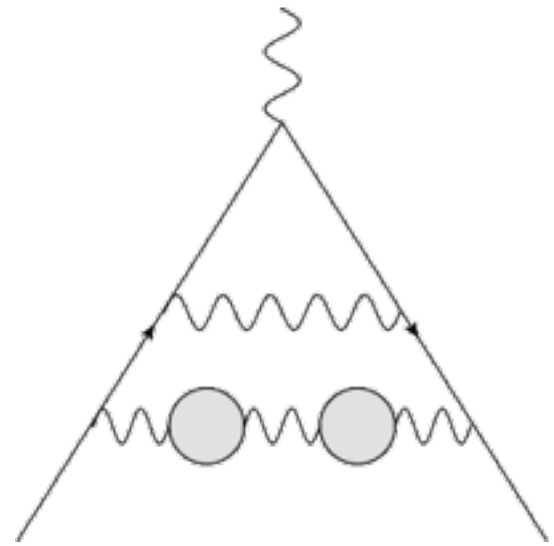
LO HVP



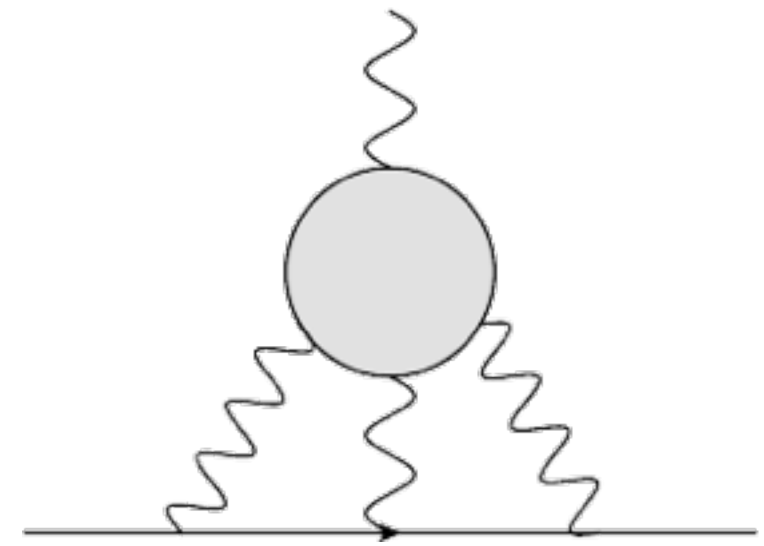
NLO HVP



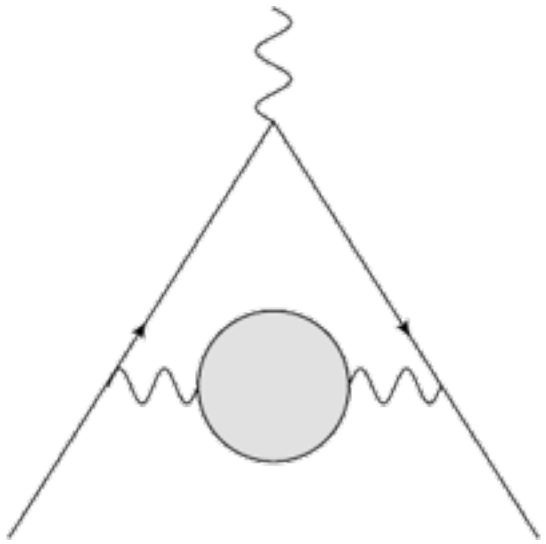
NNLO HVP



HLbL



Hadronic Vacuum Polarisation



- Currently no Standard Model prediction

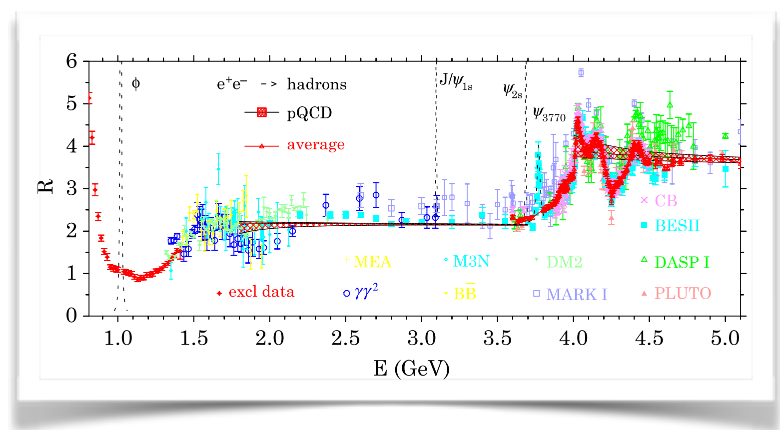
$$\Pi_{\mu\nu}(Q) = \int d^4x e^{iQ\cdot x}\langle J_\mu(x) J_\nu(0)\rangle = (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi(Q^2)$$

$$J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s \ldots$$

- needed for all Q^2

Determination from experiment:

- instead analysis of $e^+e^- \to hadrons$ cross-section

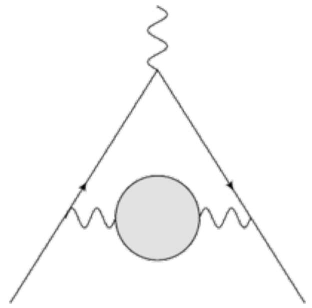


Theory determination:

$$a_\mu^{\mathrm{LO\,HAD}} = 4\alpha^2 \int_0^\infty dQ^2 f(Q^2)\left(\Pi(Q^2) - \Pi(0)\right)$$ where Q Euclidean momenta

Lautrup, Peterman, Rafael Nuovo Cim. A1 (1971) 238-242
Blum PRL.91.052001

LO HVP

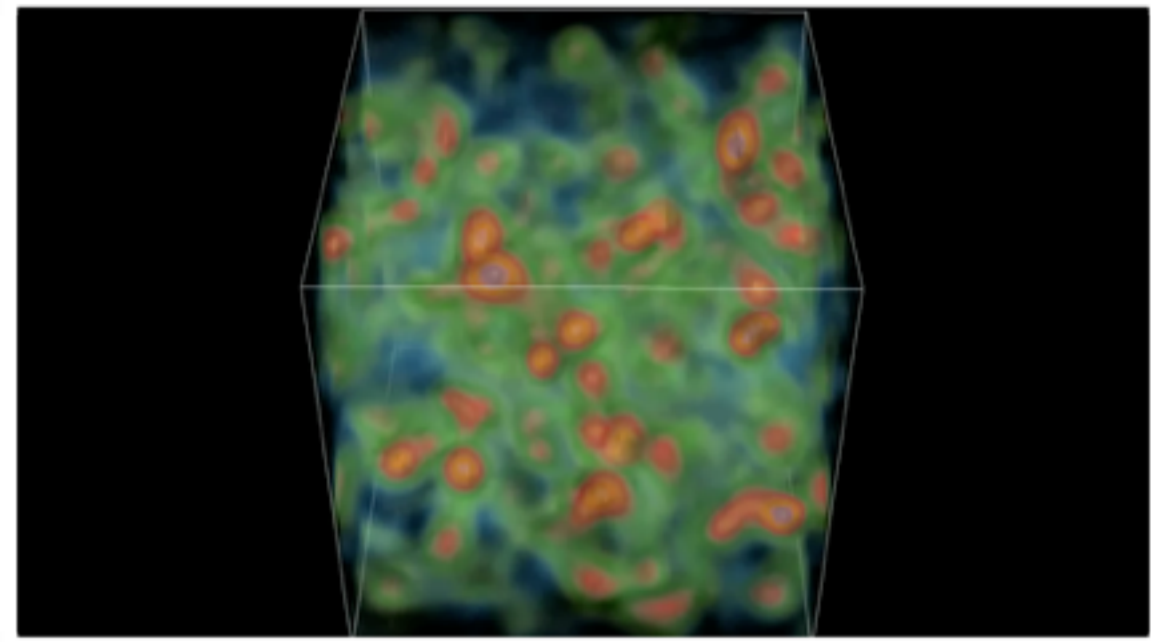


1. Simulation: compute $\Pi_{\mu\nu}(Q) = a^4 \sum_x e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0) \rangle$
2. Data analysis: determine $\Pi(Q^2)$ and integrate over Q^2

Computing $\Pi_{\mu\nu}(Q^2)$ is a text book exercise in principle — but %-level precision for a_μ is very hard

In the following:
- Status of Lattice QCD
- Major difficulties in computing a_μ
  1. Computing a_μ
  2. Finite volume effects (FVE)
  3. Statistical noise from MCMC
  4. Isospin breaking effects

State of the art of lattice QCD simulations



action density of RBC/UKQCD physical point DWF ensemble

What we can do

- simulations of QCD with dynamical (sea) *u,d,s,c* quarks with masses as found in nature ➔ $N_f = 2,\ 2+1,\ 2+1+1$
- bottom only as valence quark
- cut-off $a^{-1} \leq 4\text{GeV}$
- volume $L \leq 6fm$

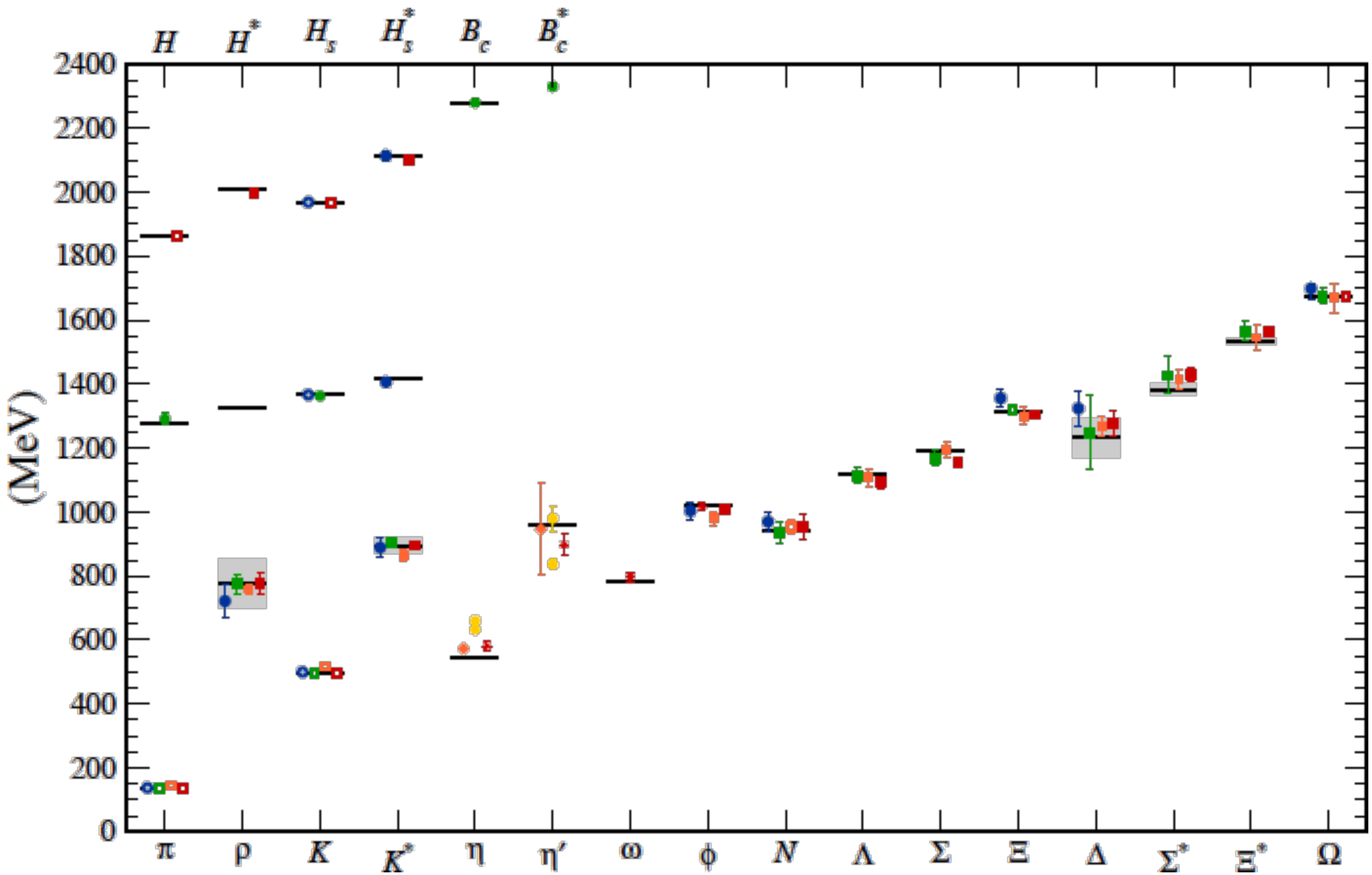
Parameter tuning

start from *educated guesses* and compute

- tune light quark mass am_l such that $\frac{am_\pi}{am_P} = \frac{m_\pi^{PDG}}{m_P^{PDG}}$
- tune strange quark mass such that $\frac{am_\pi}{am_K} = \frac{m_\pi^{PDG}}{m_K^{PDG}}$
- determine physical lattice spacing $a = \frac{af_\pi}{f_\pi^{PDG}}$

IMPORTANT:
once the QCD-parameters are *tuned* no further parameters need to be fixed and we can make fully predictive simulations of QCD

benchmark - the hadron spectrum



Kronfeld, Ann. Rev. of Nucl. Part. Sci 2012 62

HVP tensor on the lattice

$$\Pi_{\mu\nu}(Q) = a^4 \sum_x e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0)\rangle = (\delta_{\mu\nu} Q^2 - Q_\mu Q_\nu)\Pi(Q^2)$$

- For most lattice actions there exists an easily implemented conserved vector current such that $\Delta^*_\mu \langle J^{\rm cons}_\mu O\rangle = \langle \delta O\rangle$

- There are now two possible choices:
  - $O = J^{\rm cons}_\nu$ — this choice leads to a contact term
  - $O = J^{\rm local}_\nu \to \Delta^*_\mu \langle J^{\rm cons}_\mu J^{\rm local}_\nu\rangle = 0$ (local current needs to be renormalised — easy)

- $\Pi_{\mu\nu}(Q)$ from $\langle j^{\rm cons}_\mu j^{\rm loc}_\nu\rangle$ is automatically transverse up to cutoff effects which we remove by applying longitudinal projection resulting in ($p_i = 0$)

$$\Pi(Q^2) \overset{\vec{p}=0}{=} \frac{1}{Q^2}\frac{1}{3}\sum_i \Pi_{ii}(Q^2)$$

- There is also a third choice — $\langle j^{\rm loc}_\mu(x) j^{\rm loc}_\nu(0)\rangle$ use only local (not conserved) currents to construct $\Pi_{\mu\nu}$ — there will be a contact terms when $x \to 0$ which needs to be dealt with — see later

HVP - Wick contractions

$$\Pi_{\mu\nu}(Q) = a^4 \sum_x e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0)\rangle = (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi(Q^2)$$

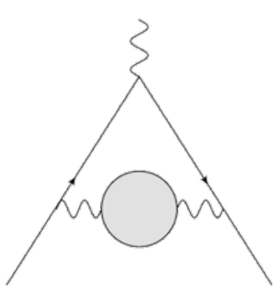
J_μ either local or (lattice) conserved

It is useful to break computation up into components:

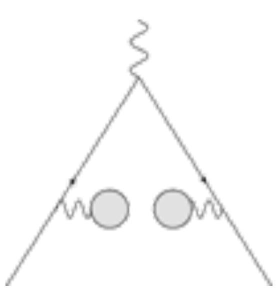
individual Wick contractions and Flavour contributions have their independent continuum and finite volume limit AJ, Della Morte *arXiv:0910.3755, JHEP11(2010)154*

allows to fine-tune simulation strategies/precision per contraction/flavour

Break up by Wick contraction



$$\Pi^{\rm conn}_{\mu\nu}(Q) = a^4 \sum_x e^{iQ\cdot x} \langle {\rm tr}\,\{\gamma_\mu S(x,0)\gamma_\nu S(0,x)\}\rangle$$ by far dominant part



$$\Pi^{\rm disc}_{\mu\nu}(Q) = a^4 \sum_x e^{iQ\cdot x} \langle {\rm tr}\,\{\gamma_\mu S(x,x)\}\,{\rm tr}\,\{\gamma_\nu S(0,0)\}\rangle$$ small correction

HVP - Wick contractions

Analytical considerations for Wick contractions:

- Disconnected contribution zero in SU(3) limit

- PQChPT NLO: $$\frac{\Pi^{\rm disc}_{\mu\nu}(Q)}{\Pi^{\rm conn}_{\mu\nu}(Q)} = -\frac{1}{10}$$ confirmed at NNLO

AJ, Della Morte JHEP11(2010)154

Bijnens, Relefors arXiv:1609.01573

Ignores ρ contribution to VP. $\pi\pi$ contribution estimated to be ~10% (model), would reduce to -1/10*0.1 = 1% effect HPQCD PhysRevD.93.074509 (2016)

→ **Can be more relaxed about precision goal for disconnected contribution**

Break up by flavour

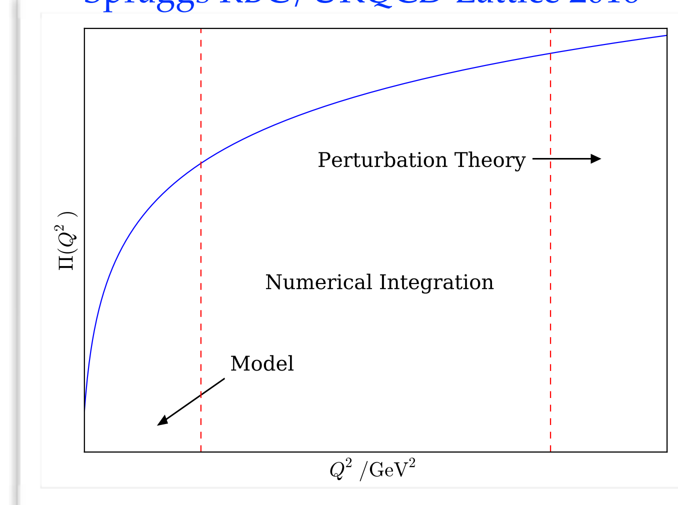
Connected up/down	— strange	— charm contributions
90%	8%	2%

- Unfortunately high precision easier for heavy flavour contribs
- Disconnected contributions mix flavour at source and sink

From the HVP to a_μ

$$a_\mu^{\text{LO HAD}} = 4\alpha^2 \int_0^\infty dQ^2 f(Q^2)\left(\Pi(Q^2) - \Pi(0)\right)$$

Spraggs RBC/UKQCD Lattice 2016



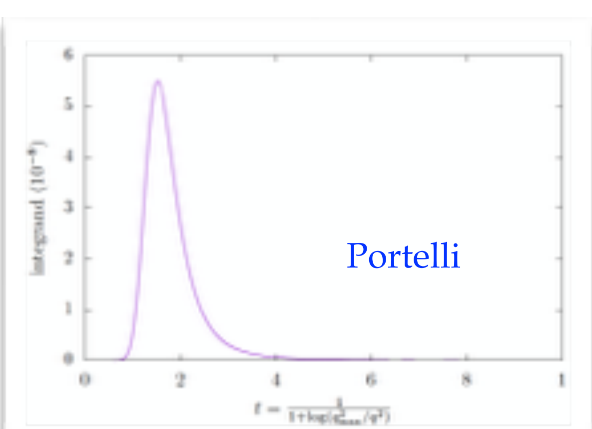
There are essentially three different ways for extracting a_μ:

- ***Traditional analysis — fits to HVP***
  - fit ansätze studied in detail Aubin et al. PhysRevD.86.054509
  - low-Q² problem $\Pi(Q^2) = \Pi_{\mu\nu}(Q)/(\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)$
- ***Time moments*** HPQCD PhysRevD.89.114501
  - zero momentum projected correlator: $G(t) = a\sum_{\vec{x}} \langle j_i(t,\vec{x}) j_i(0,\vec{0})\rangle$

$$G_{2n} \equiv a^4 \sum_t t^{2n} G(t) = (-1)^n \frac{\partial^{2n}}{\partial Q^{2n}} Q^2 \hat{\Pi}(Q^2)|_{Q^2=0}$$

$$\hat{\Pi}(Q^2) = \sum_{j=1}^{\infty} Q^{2j}\Pi_j$$

$$\Pi_j = (-1)^{j+1}\frac{G_{2j+2}}{(2j+2)!}$$

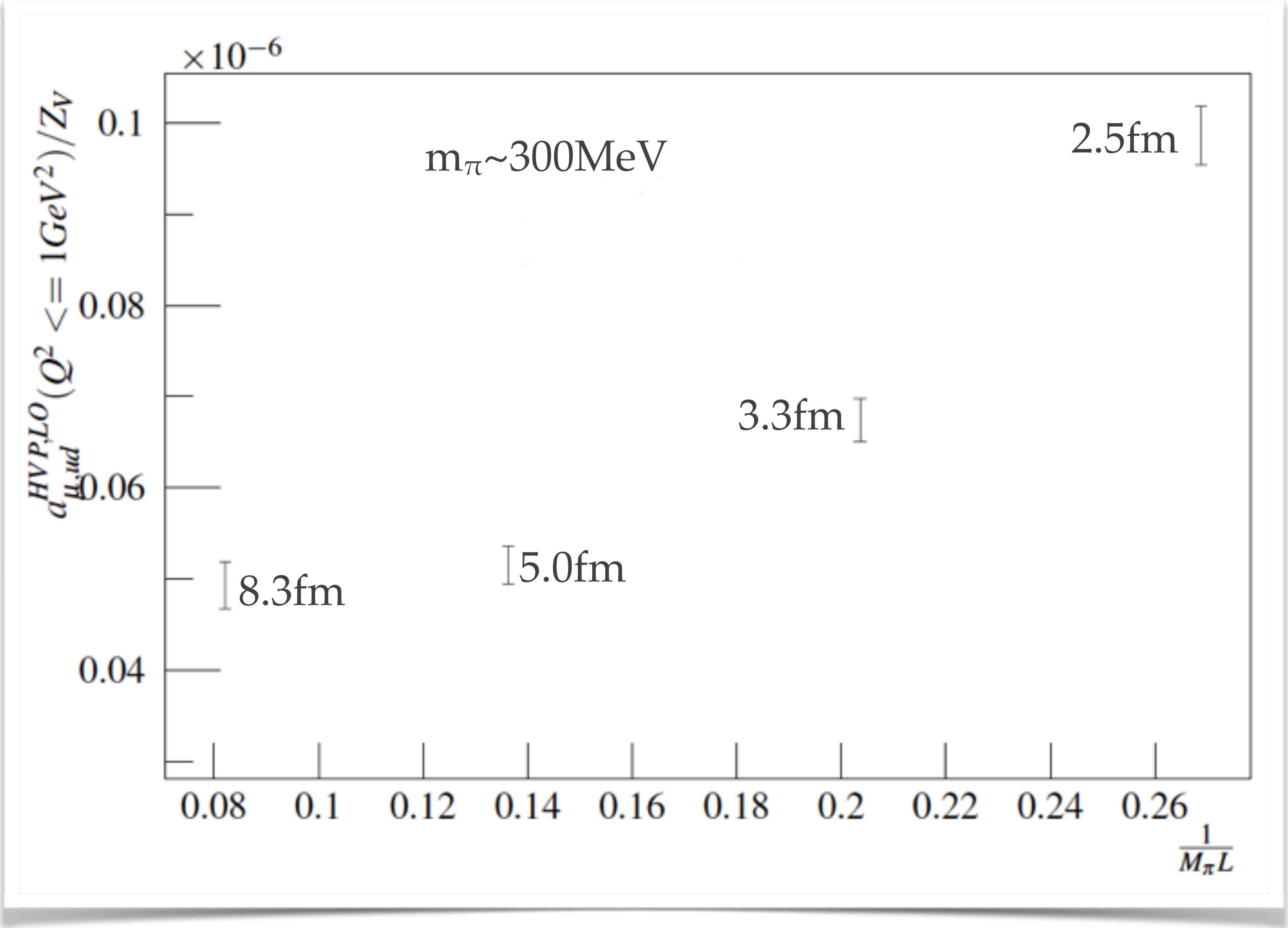


- ***Sine Cardinal interpolation*** — use Fourier transform with continuous momenta

Feng et al. PhysRevD.88.034505, Bernecker, Meyer epja/i2011-11148-6 , Portelli, Del Debbio in preparation

Finite Volume Effects

BMW's finite volume scaling study for a_μ



BMW arXiv:1502.02172

Finite Volume Effects in ChPT

Aubin et al. PhysRevD.93.054508

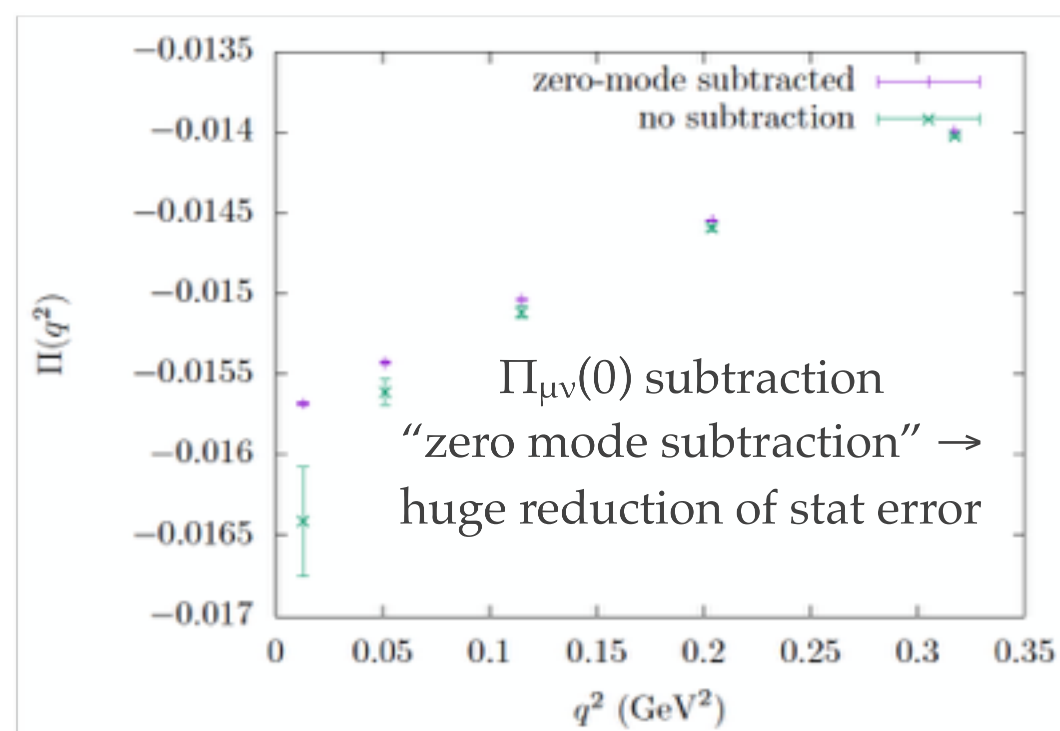
- In finite volume with L≠T, rotation group broken down to group of cubic rotations
- Finite volume effects in ChPT as per irreducible representation ($A_1,\ A_1^{44},\ T_1,\ T_2,\ E$)

Results:

- $\Pi_{\mu\nu}(0) \neq 0$ in finite volume (known before) — but subtracted VP tensor

$$\overline{\Pi}_{\mu\nu}(Q) = \Pi_{\mu\nu}(Q) - \Pi_{\mu\nu}(0)$$

  by an order of magnitude closer to infinite-volume points BMW arXiv:1502.02172
  - confirms previous BMW study
  - further benefit: $\Pi_{\mu\nu}(0)$ and $\Pi_{\mu\nu}(Q^2)$ highly correlated in MCMC data, *subtracting zero* significantly reduces stat. error

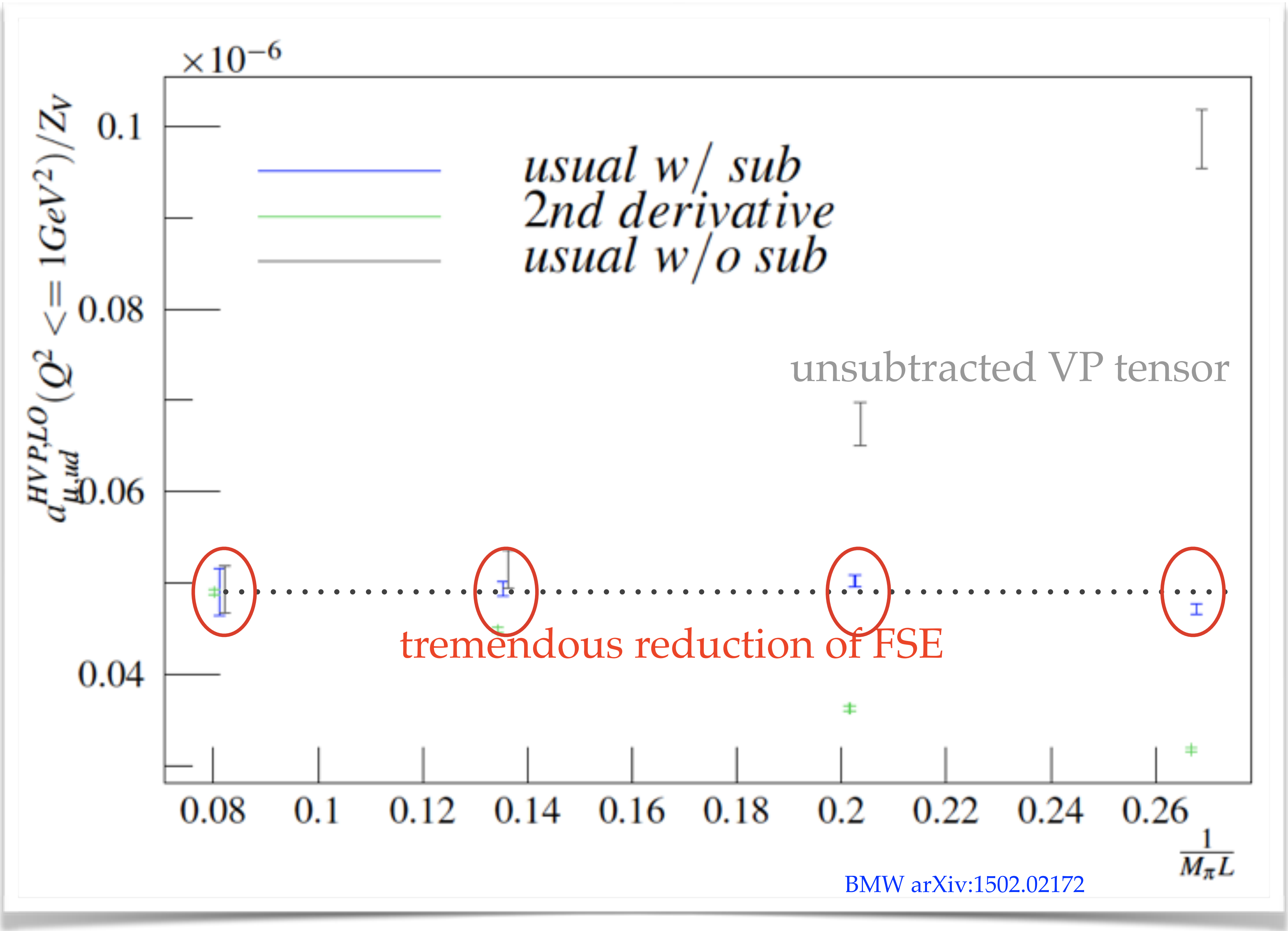


- even for $m_\pi L > 4$ FSE can be of order 10%

- *Conservative* estimate of finite volume errors: infinite volume result lies between result for two different irreps ($A_1,\ A_1^{44}$)

Finite Volume Effects

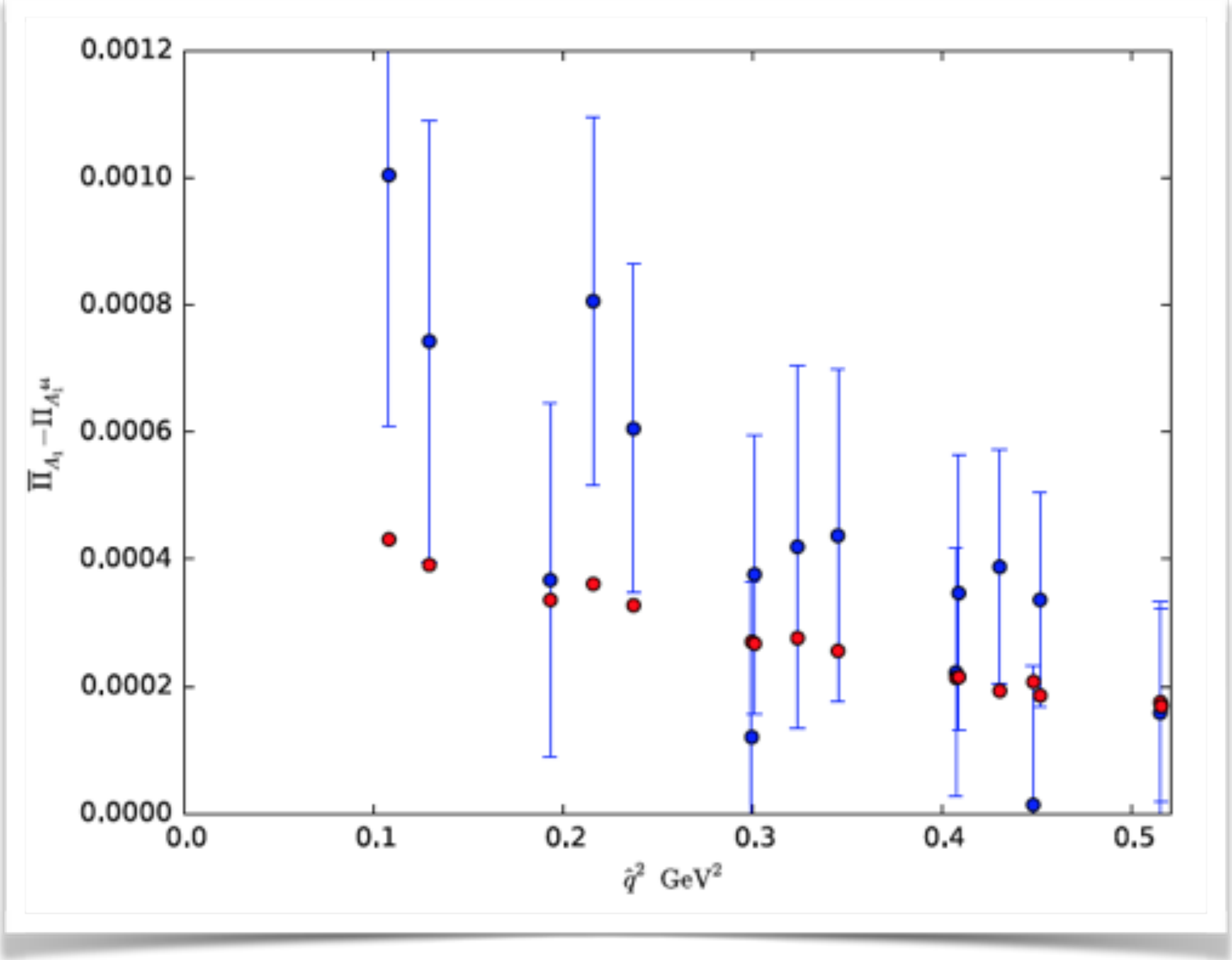
data confirms small FVE for subtracted VP tensor $\overline{\Pi}_{\mu\nu}(Q) = \Pi_{\mu\nu}(Q) - \Pi_{\mu\nu}(0)$



Finite Volume Effects

Aubin et al. PhysRevD.93.054508

Does ChPT agree with data?
ChPT only properly describes 2π contribution to FVE (not the ϱ resonance contrib.)
→ consider differences of finite volume effects, e.g. different irreps: A_1-A_1^{44}
(differences of finite volume effects will be dominated by 2π effects)



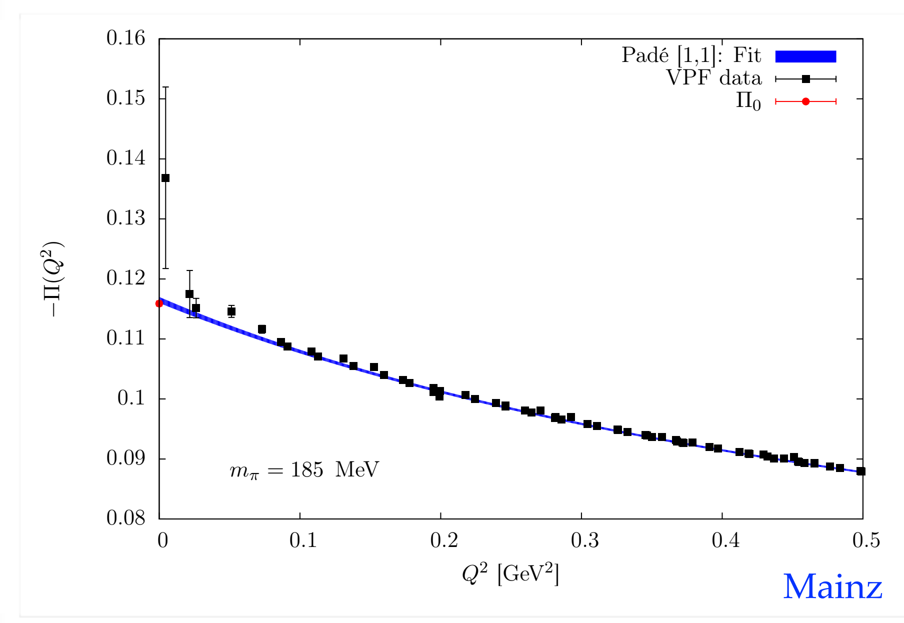
Summary finite volume effects:

- good agreement between eff. theory and lattice data for differences of FVE
- can define estimate of FVE
- hope is that ChPT can be used to control FVE at 1% level but further testing necessary

Signal-To-Noise

$$\Pi_{\mu\nu}(t,\vec{Q}) = \int d^4x e^{i\vec{Q}\vec{x}} \langle J_\mu(t,\vec{x}) J_\nu(0) \rangle$$

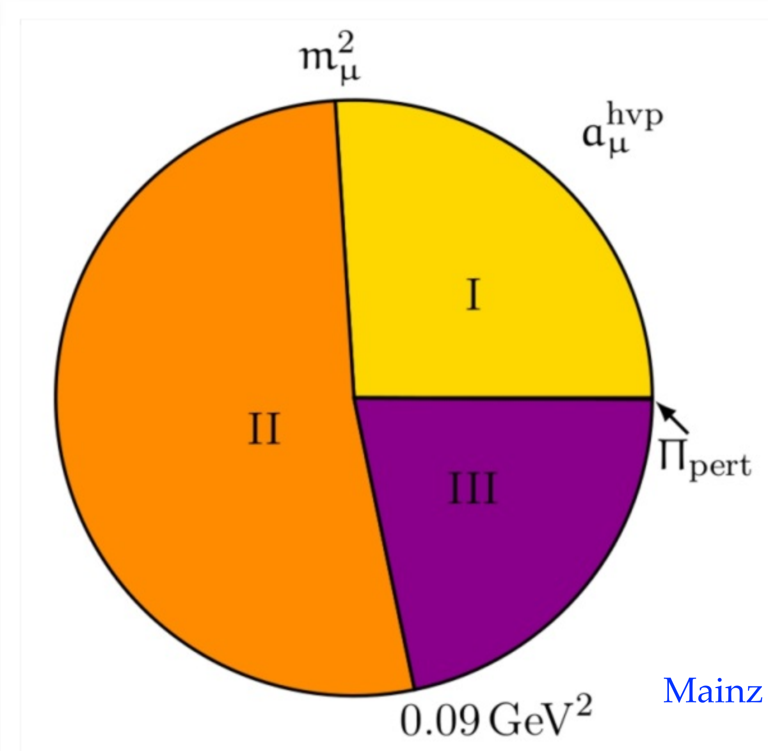
- Correlation function easy to compute but signal-to-noise deteriorates for small momenta. This is expected due to the understood exponential deterioration of the signal-to-noise ratio at large distance in the vector correlator



- This is really bad since the Kernel of

$$a_\mu^{\mathrm{LO\,HAD}} = 4\alpha^2 \int_0^\infty dQ^2 f(Q^2)\left(\Pi(Q^2) - \Pi(0)\right)$$

receives dominant contribution from low Q^2 region



Signal-To-Noise

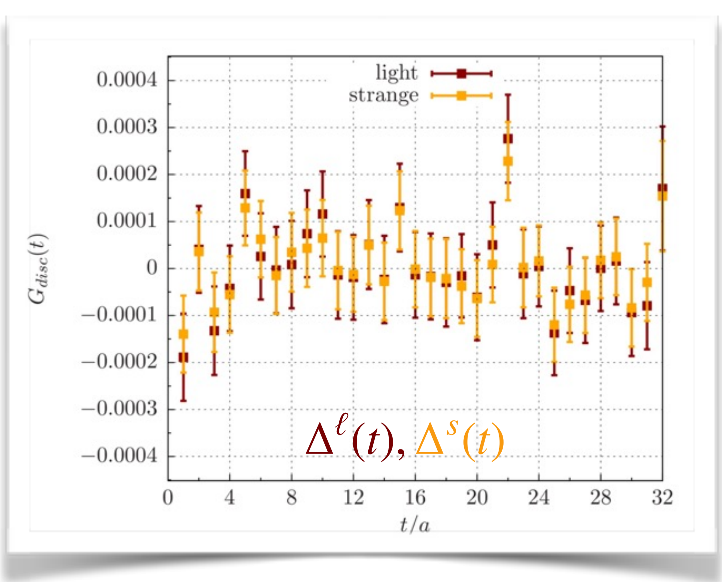
Example for how we are currently dealing with signal-to-noise issue: RBC/UKQCD's computation of **quark-disconnected contribution** on Domain Wall Fermion ensembles with physical sea pions RBC/UKQCD PhysRevLett.116.232002

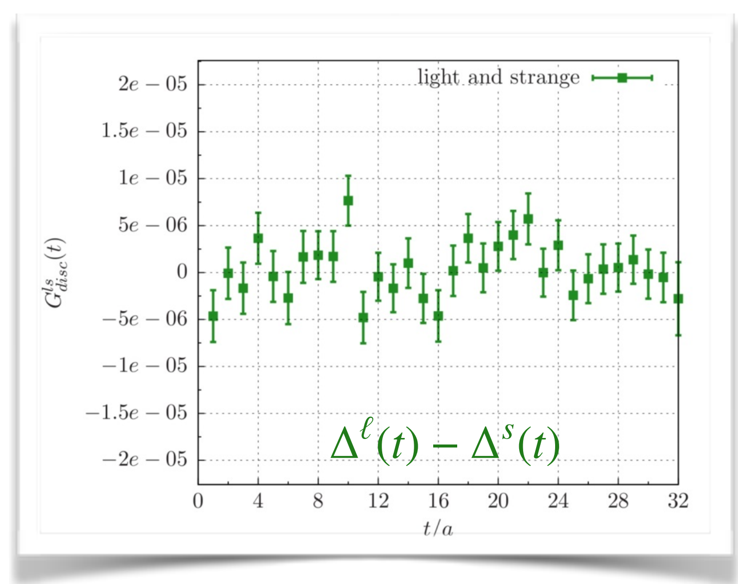
consider disconnected correlator:

$$C(t) = \frac{1}{3V}\sum_{i,t'}\langle \mathcal{V}_i(t)\mathcal{V}_i(0)\rangle \text{ and } \mathcal{V}_i = \frac{1}{3}\left(\mathcal{V}_i^{u/d} - \mathcal{V}_i^{s}\right)$$

$$\mathcal{V}_i^f(t) = \sum_{\vec{x}} \mathrm{ImTr}\left(S^f(x,x)\gamma_i\right)$$

- Mainz group observed: stat. fluctuations of s- and u/d quarks anti-correlated
  →statistical error in difference of s and l quarks cancel Gülpers Lattice 2014





Signal-To-Noise

Let's go to *time-momentum representation* ($\vec{Q}=0$)

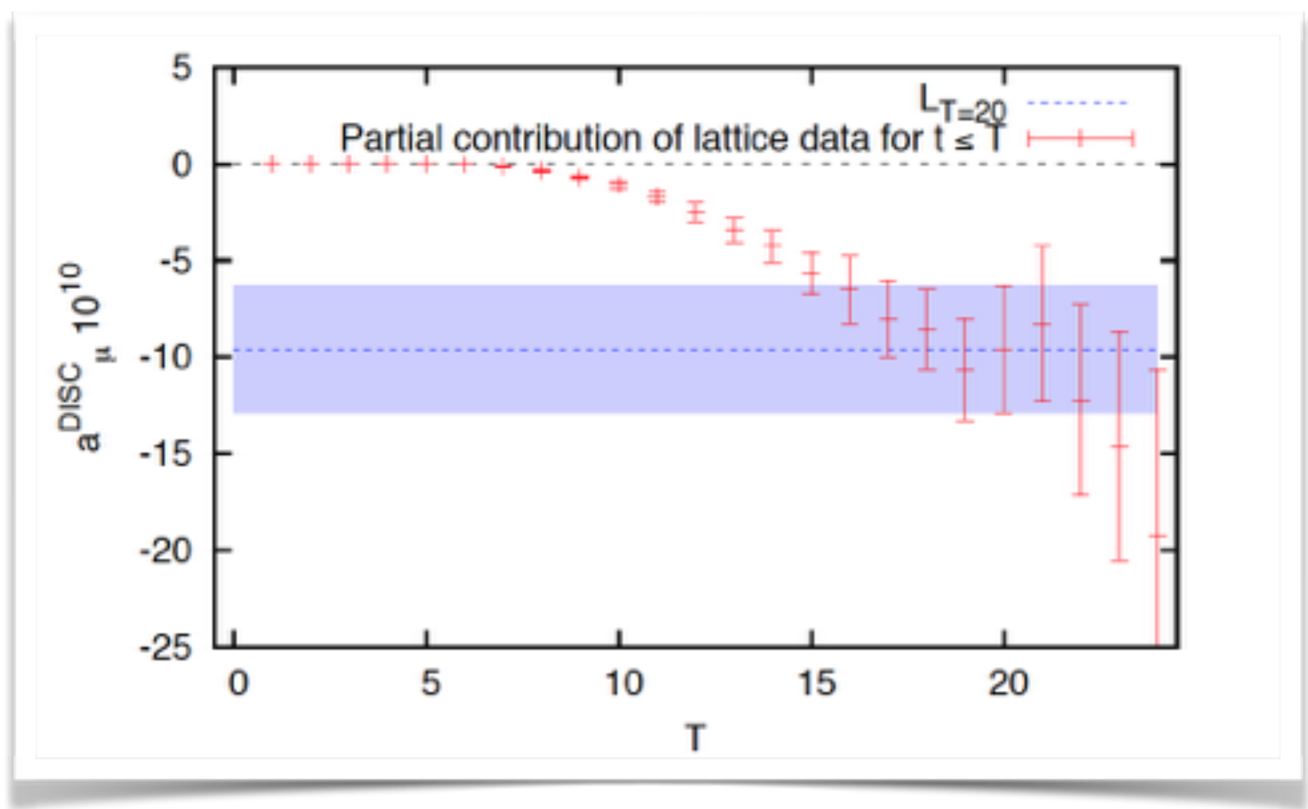
$$\Pi(Q^2)-\Pi(0)=\frac{1}{3}\sum_t\left(\frac{\cos(Q_t t)-1}{q^2}+\frac{1}{2}t^2\right)G(t) \qquad a_\mu^{\mathrm{LO\,HVP}}=\sum_{t=0}^{\infty}w(t)G(t)$$

Bernecker, Meyer epja/i2011-11148-6

Consider partial sum up to time-extent T

$$L_T=\sum_{t=0}^{T}w(t)G(t)$$

- Signal-To-Noise issue clearly visible
- G(t) consistent with zero for t ≥ 15



Idea: use $G(t)=\begin{cases}G(t)^{\mathrm{data}}, & t\le t^{\mathrm{cut}}\\ G(t)^{\mathrm{model}}, & t>t^{\mathrm{cut}}\end{cases}$

Signal-To-Noise

RBC/UKQCD PhysRevLett.116.232002

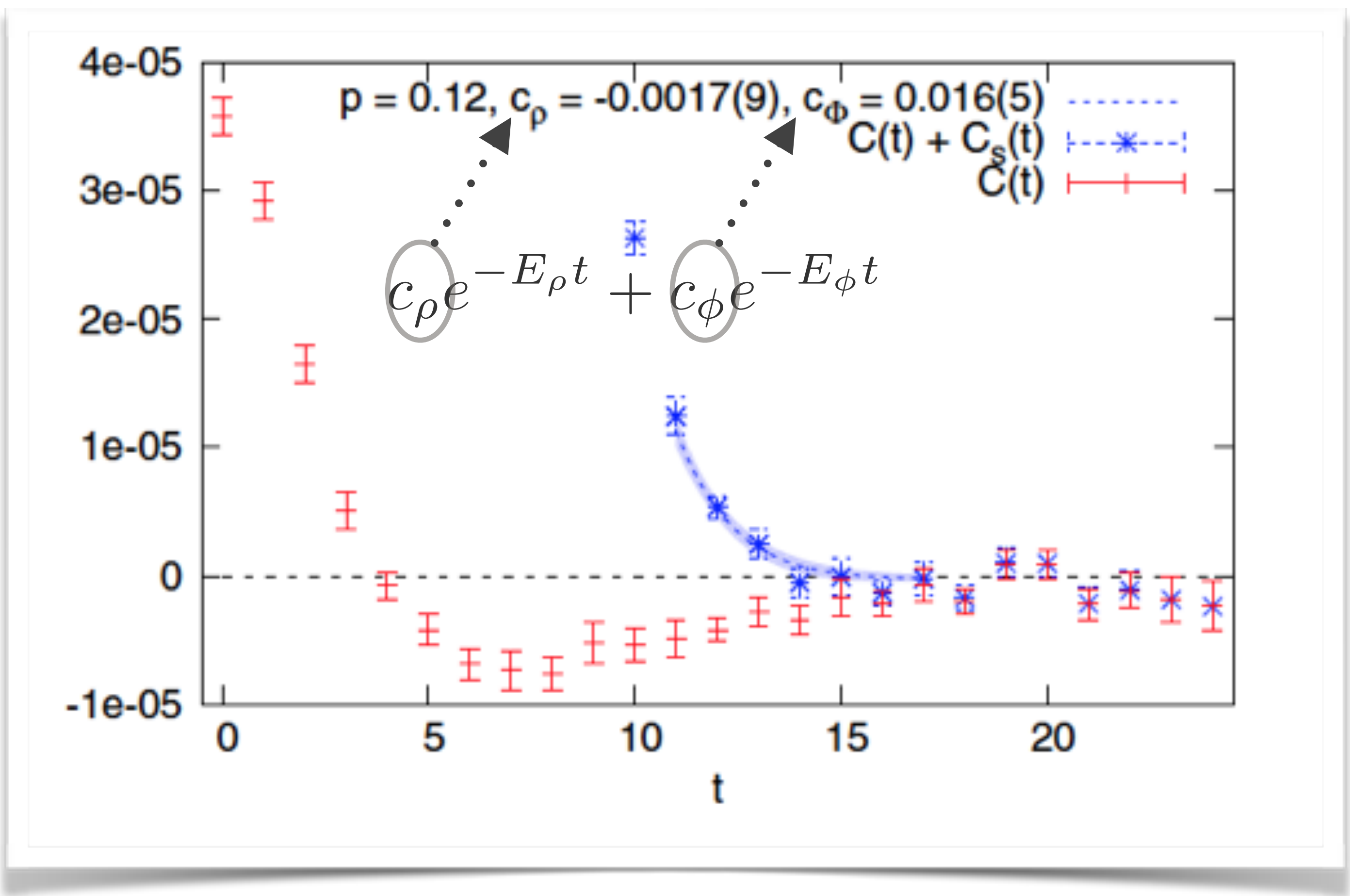
- using isospin and flavour algebra we can write the light-disconnected contribution as a correlation function with a continuum and infinite volume limit

$$\langle V_\mu^{uu} V_\nu^{uu} \rangle - \langle V_\mu^{ud} V_\nu^{du} \rangle$$

AJ, Della Morte JHEP11(2010)154

- not possible for strange contribution but consider instead

$$\langle (V_\mu^{uu} - V_\mu^{ss}) (V_\nu^{uu} - V_\nu^{ss}) \rangle - \langle V_\mu^{ud} V_\nu^{du} \rangle = C(t) + C_s(t) = \sum_m c_m e^{-E_m t}$$



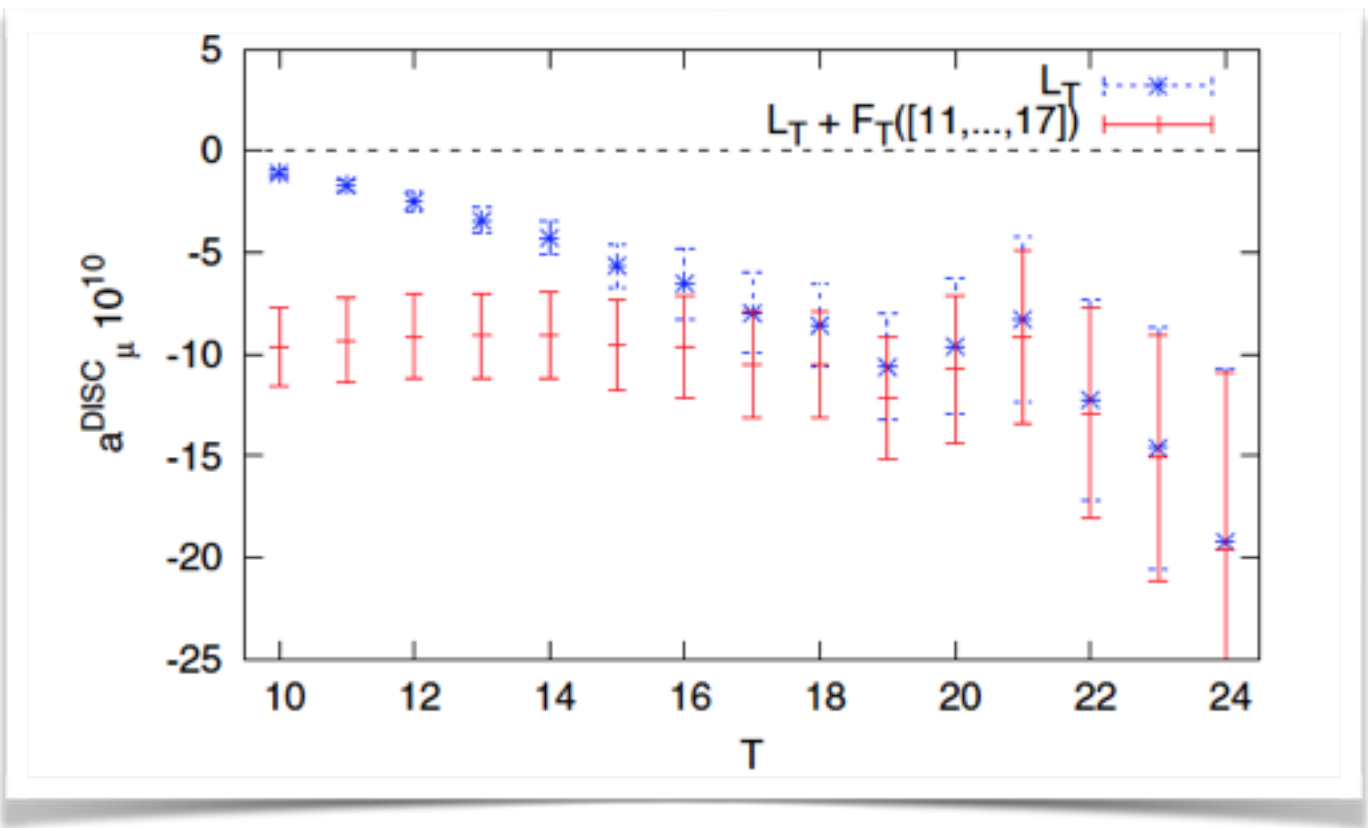
Signal-To-Noise

$$L_T = \sum_{t=0}^{T} w(t) G(t)$$

$$F_T(r) = \sum_{t=T+1}^{t_{\max}} w(t) \left(c_\rho^r e^{-E_\rho t} + c_\phi^r e^{-E_\phi t} - C_s(t) \right)$$

strange quark connected

- E_ρ, E_ϕ from experiment, c_ρ, c_ϕ from fit
- **central value for a^{DISC} from L_T**
- **systematic error due to cut from F_T**



final result from T=20

$a^{DISC} = -9.6(3.3)\times 10^{-10}$

Systematics:

- Finite T effects
- Finite volume errors ($\pi\pi$ in ChPT)
- Cutoff effects
- Variations in fit range to $C+C_s$:

$$a^{DISC} = -9.6(3.3)(2.3)\times 10^{-10}$$

This is our result ($N_f = 2+1$) for physical pion mass!!!

status LO HVP

a_μ x 10^{10}	HPQCD (arXiv:1601.03071)	RBC/UKQCD (JHEP 1604 (2016) 063 arXiv:1602.01767 arXiv:1512.09054)
light	598(11)	work in progress
strange	53.4(6)	52.4(2.1)
charm	14.4(4)	work in progress
disconnected	0(9)	–9.6(3.3)(2.3)
all	666(6)(12)	—
SM OK exp all	720(7)	720(7)

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} = 720(7)$$

- strange, charm and bottom sufficiently precisely known
- getting the disconnected in full LQCD was a big achievement (previously considered show stopper)

- first results (HPQCD) indicate tension confirmed

Need to concentrate on:

- stat. error on light contribution
- strong and elm. isospin breaking effects

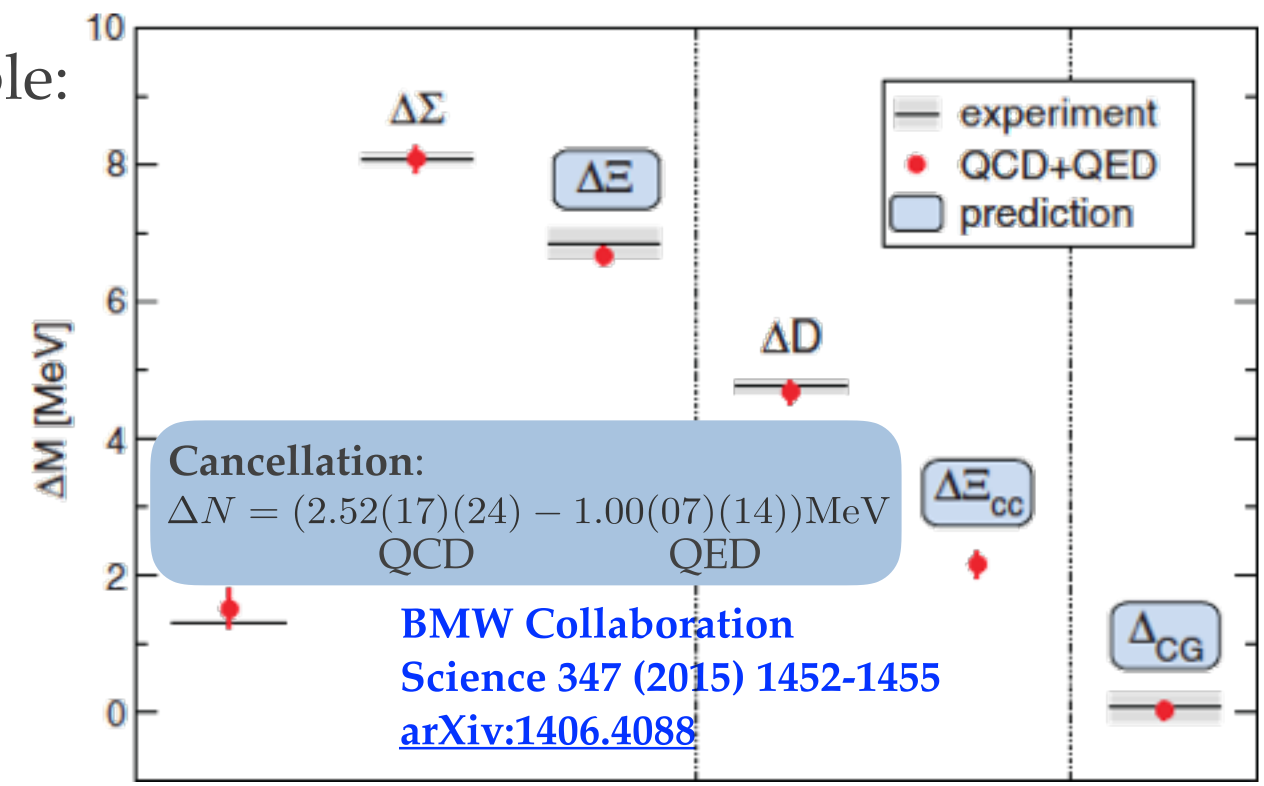
Isospin Breaking Effects

- Most current simulations $N_f = 2+1(+1)$ flavour

  $m_u = m_d$, α_{EM}

- QED effects in HVP expected to be ~1% — needs to be taken seriously

- L(QED+QCD) has become quite fashionable:
  - post/predicting hadron spectra/mass splittings
  - including QED for matrix elements theoretically/technically challenging — IR divergences (Bloch-Nordsieck)
    Carrasco et al. PRD 91 074506 (2015) **arXiv:1502.00257**



- a_μ is special — no IR divergences

should be doable modulo finite volume effects due to the photon (later)

Isospin Breaking Effects

Stochastic method Duncan PhysRevLett.76.3894

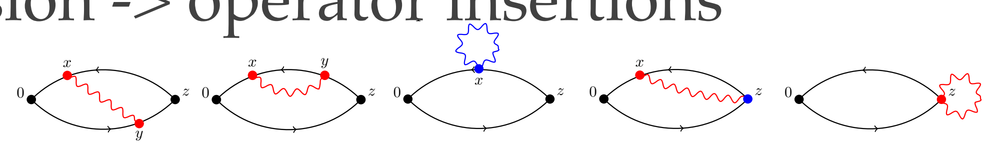
- QCD+quenched QED
- generate U(1) gauge configs
- Promote SU(3) gauge links to U(3)

$$U_\mu^{U(3)}(x) = e^{iq_{\text{em}}A_\mu(x)}U_\mu^{SU(3)}(x)$$

- γ zero-mode subtracted
- Feynman or Coulomb gauge

Perturbative method Rome123 PhysRevD.87.114505

- expand QCD+QED path integral in α, drop sea quark contribution
- expansion -> operator insertions
- *O(α):*

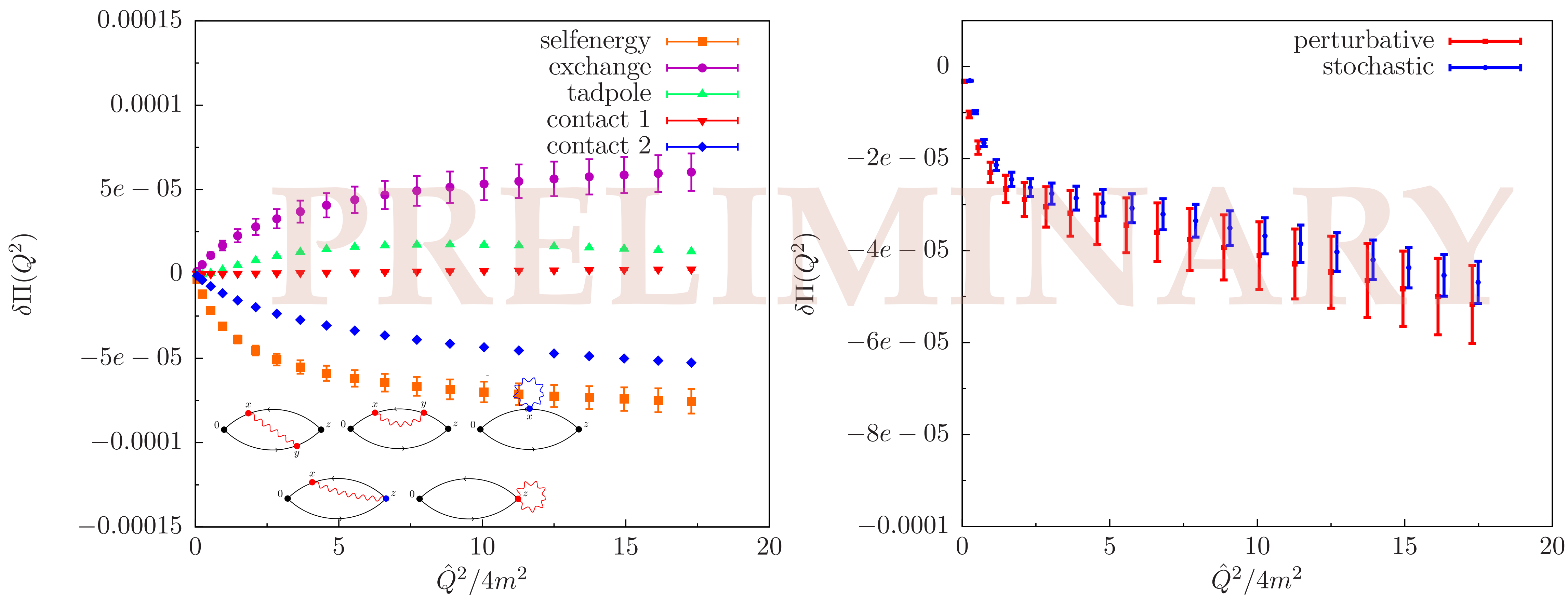


- insert Feynman/Coulomb gauge photon propagator

The Southampton group is computing isospin breaking effects using both techniques (see also **Harrison's and Gülper's** talks at Lattice 2016)

Isospin Breaking Effects

preliminary results for the finite volume isospin breaking
perturbative vs. stochastic approach
QED+free QCD



See also RBC/UKQCD's Vera Gülper and James Harrison
talks at Lattice 2016

Isospin Breaking Effects

preliminary results for the finite volume isospin breaking
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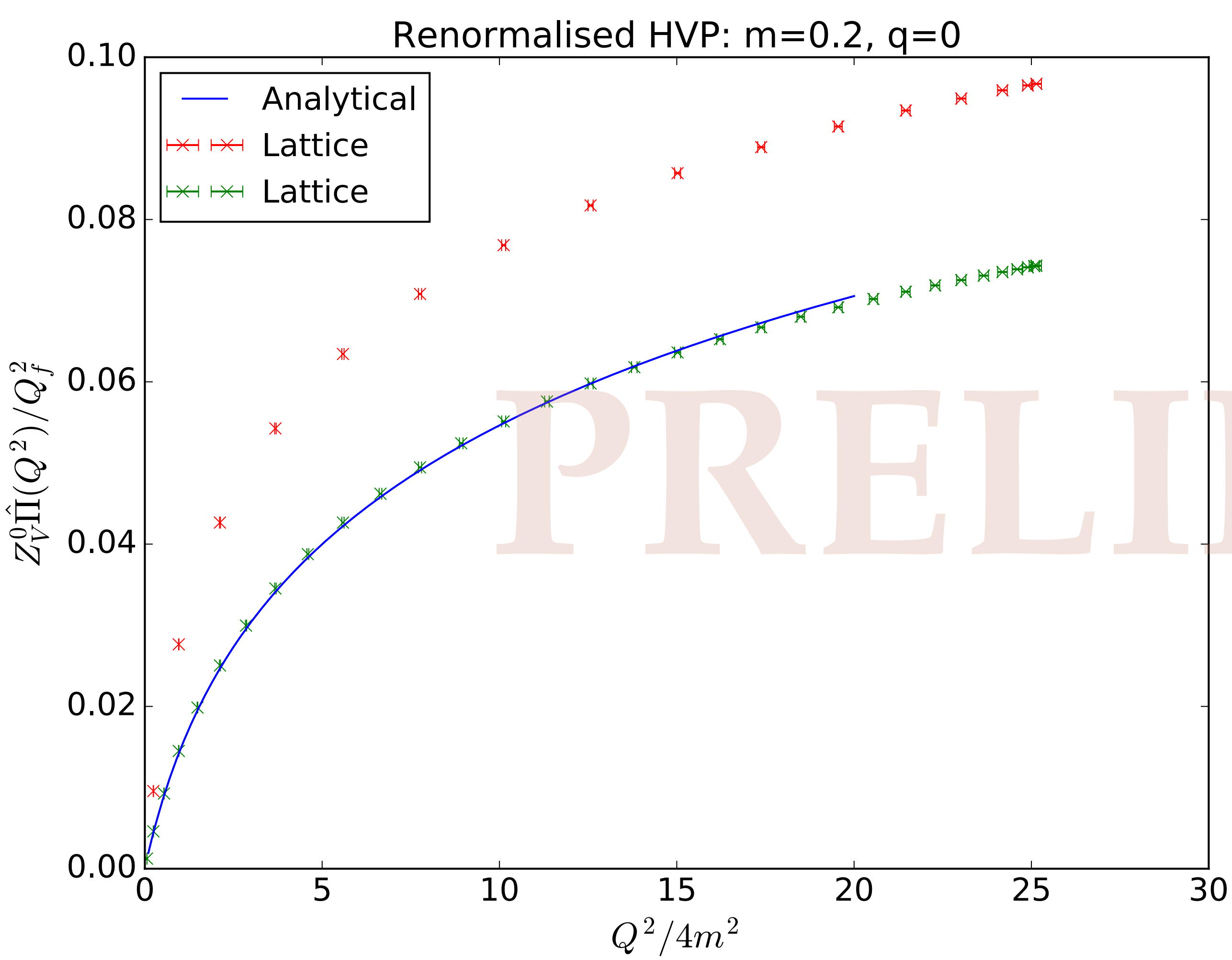
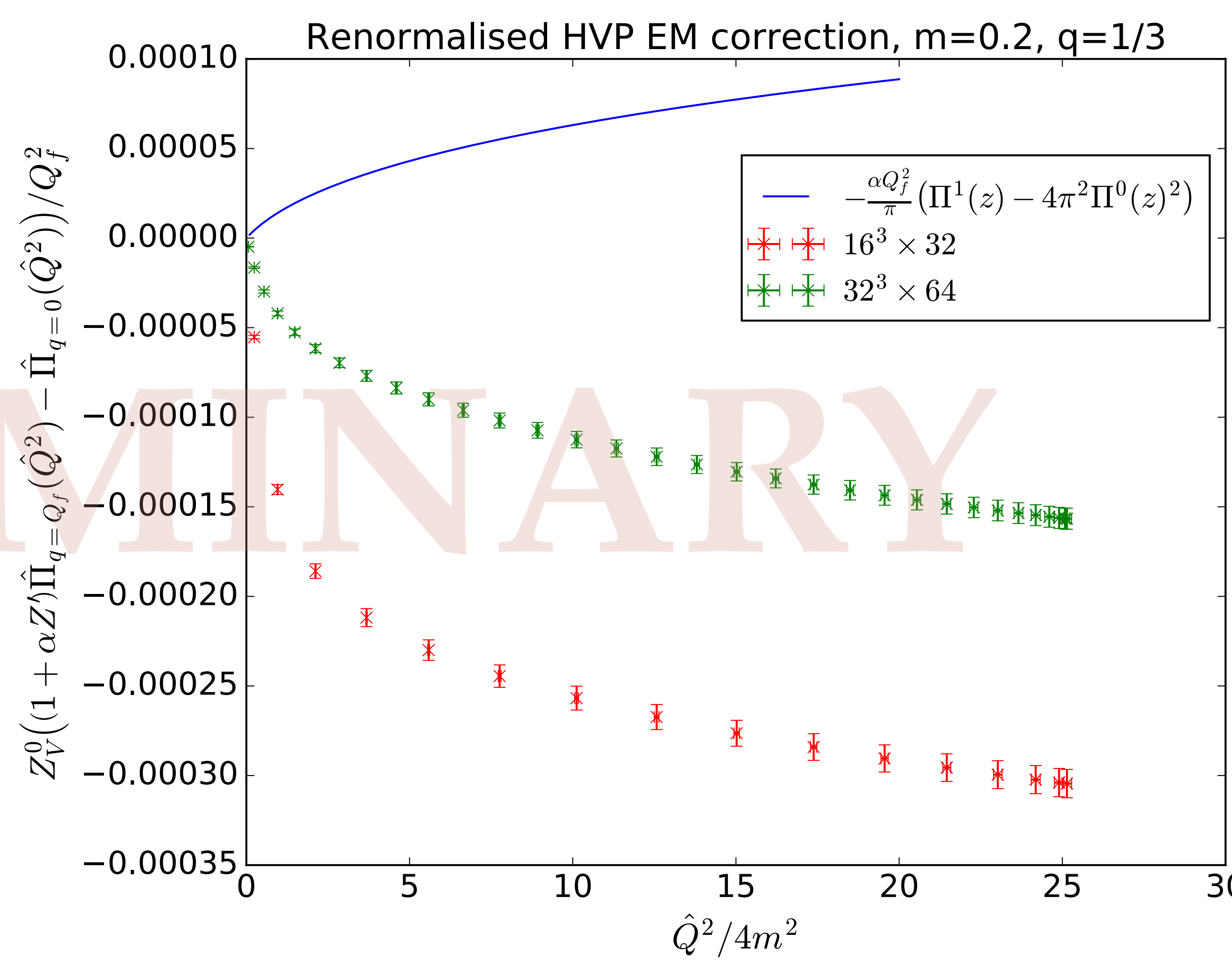
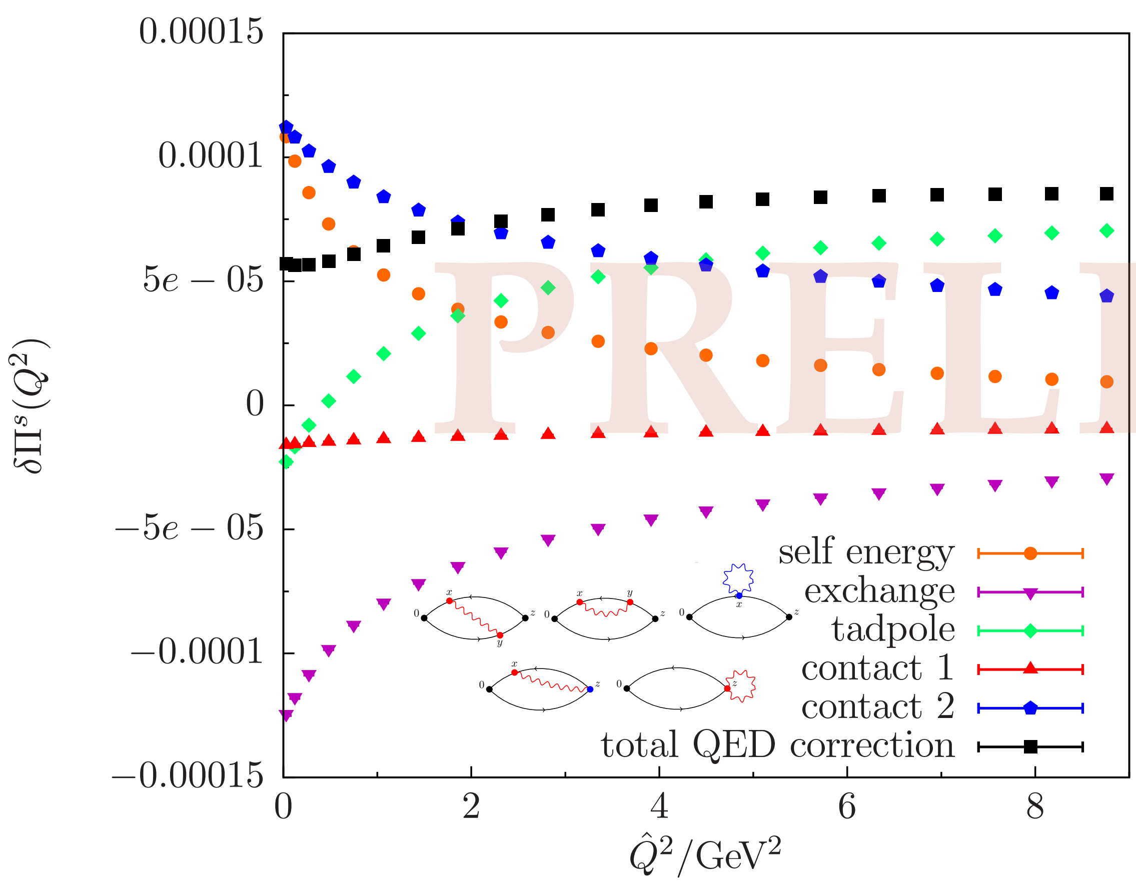


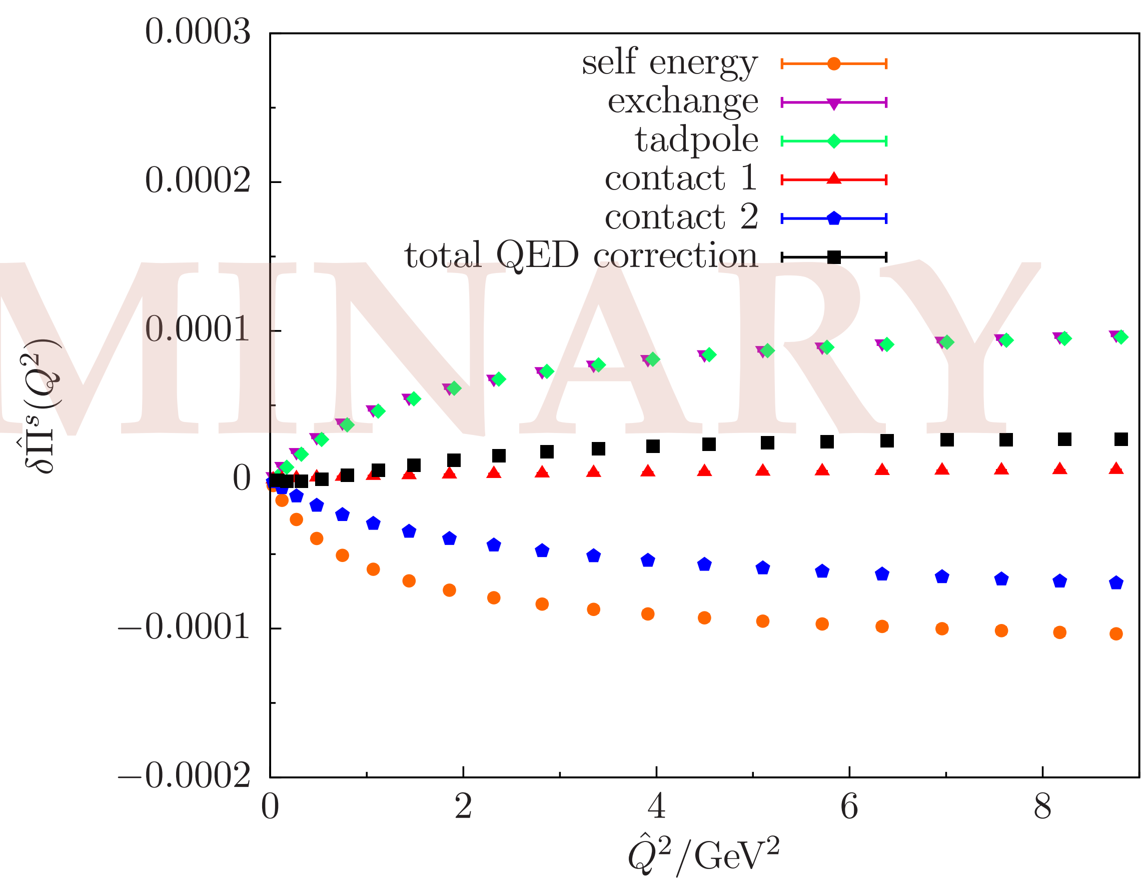


Tremendous Finite Volume effects — not unexpected but needs to be studied in detail

Isospin Breaking Effects

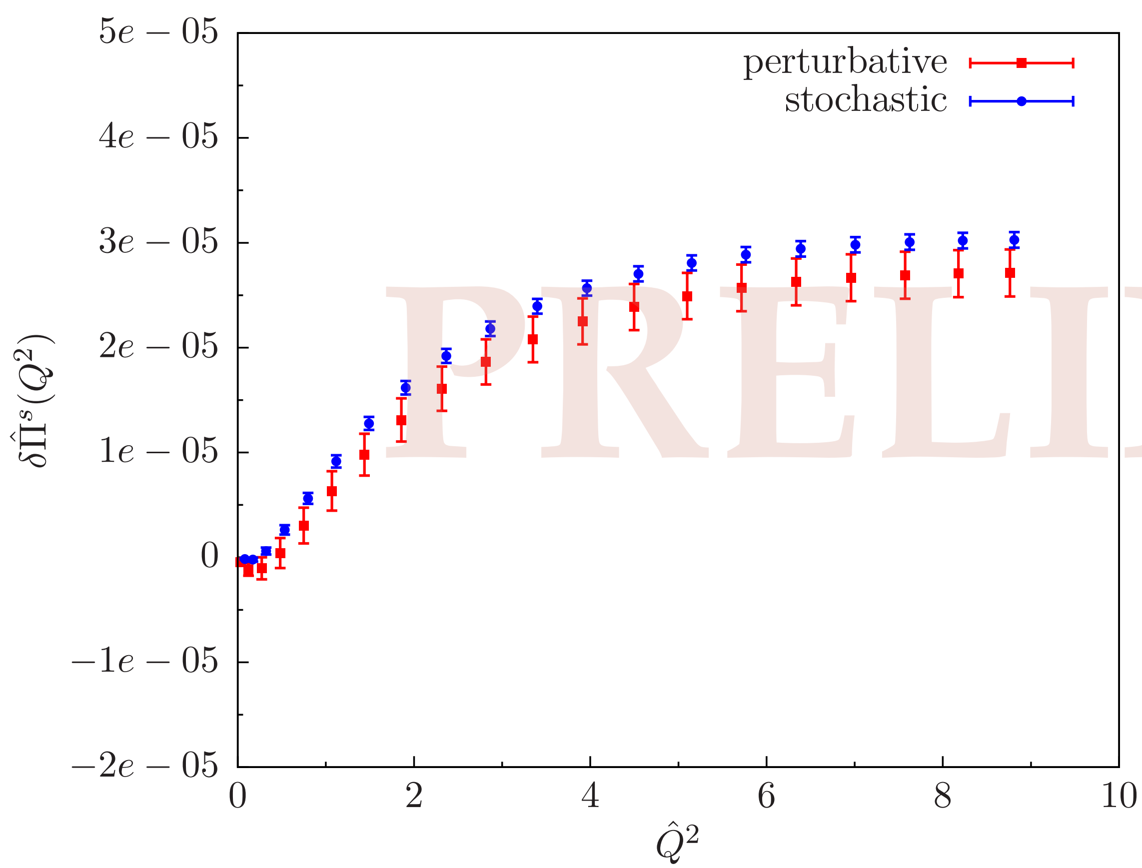
preliminary results for the finite volume isospin breaking
the strange contributions to the HVP, perturbative vs. stochastic approach
QED+QCD (DWF, 24^3, 1.7GeV, m_π=330MeV)

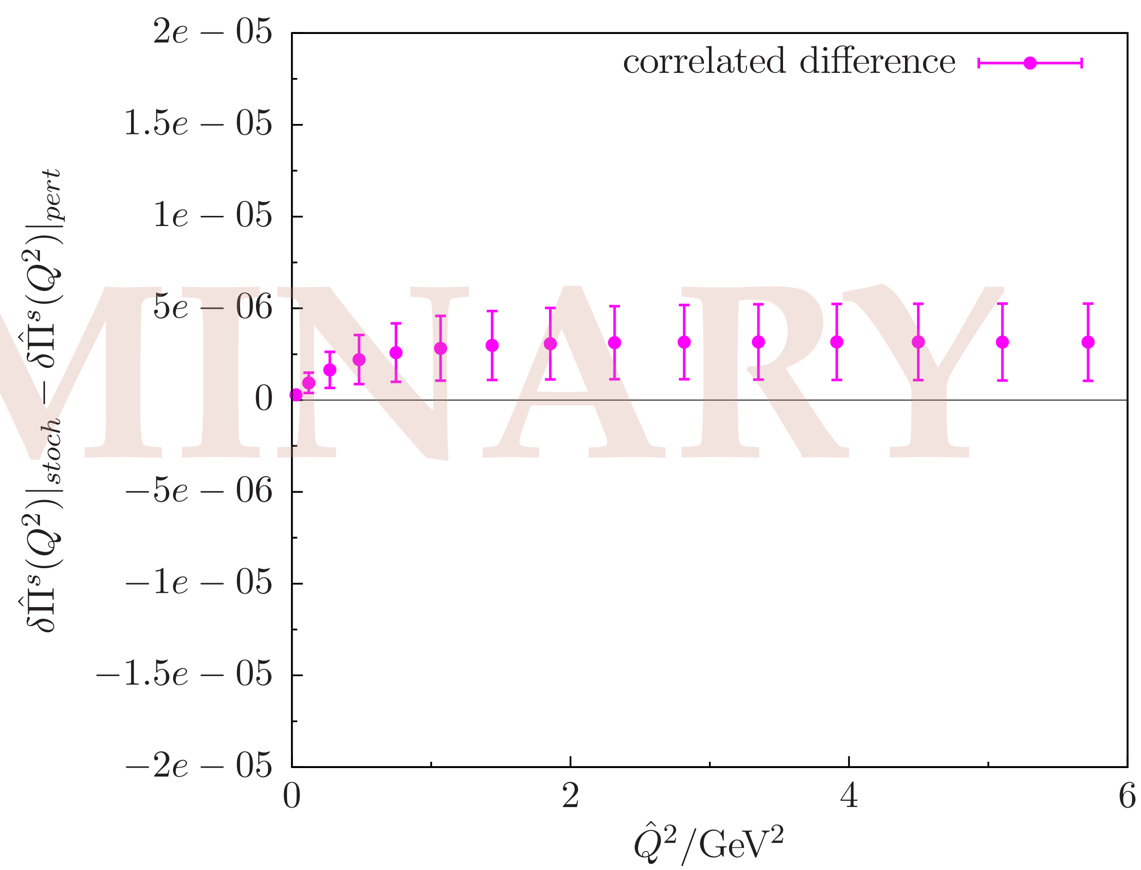




Isospin Breaking Effects

very preliminary results for the finite volume isospin breaking
the strange contributions to the HVP, perturbative vs. stochastic approach
QED+QCD (DWF, 24^3, 1.7GeV, m_π=330MeV)





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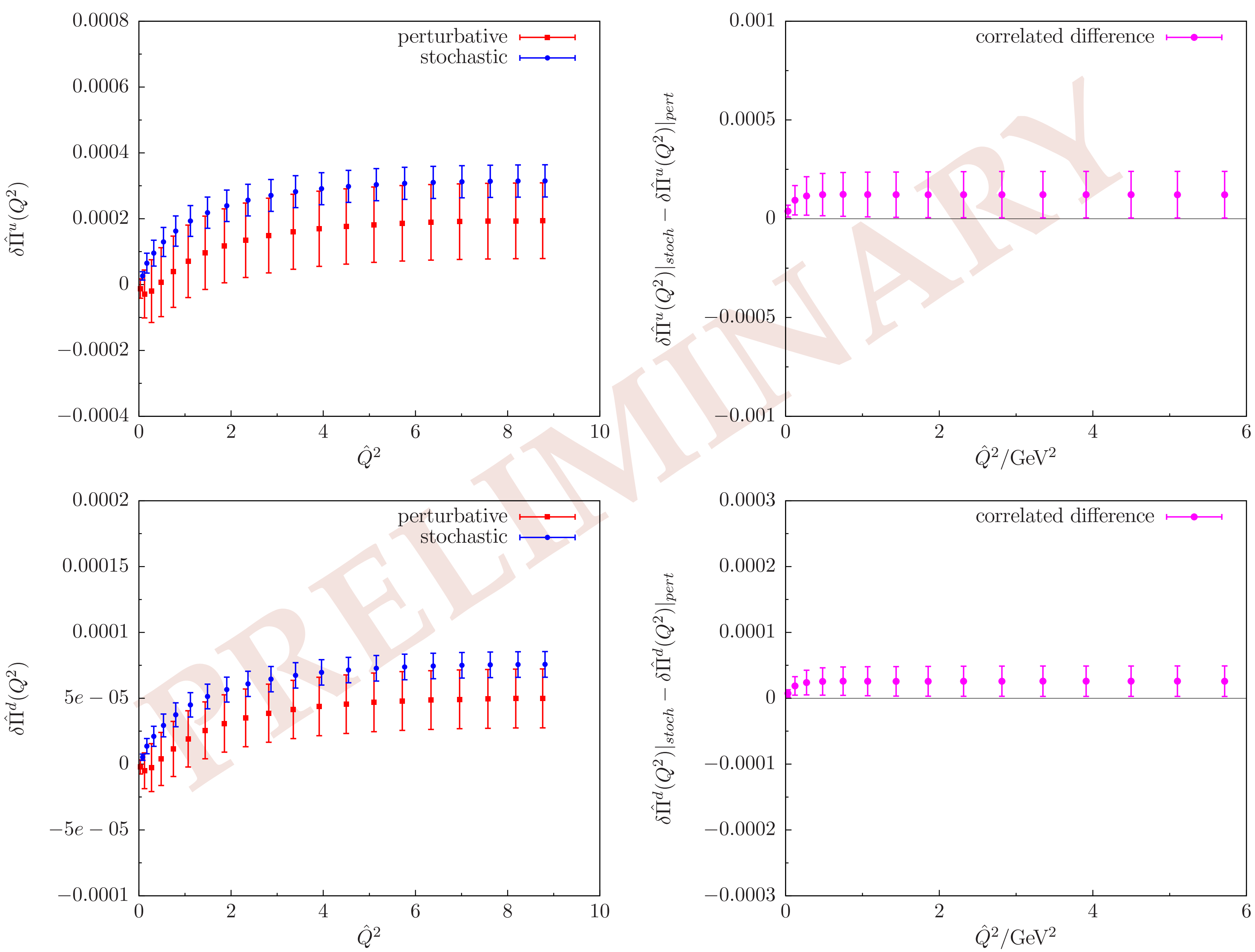
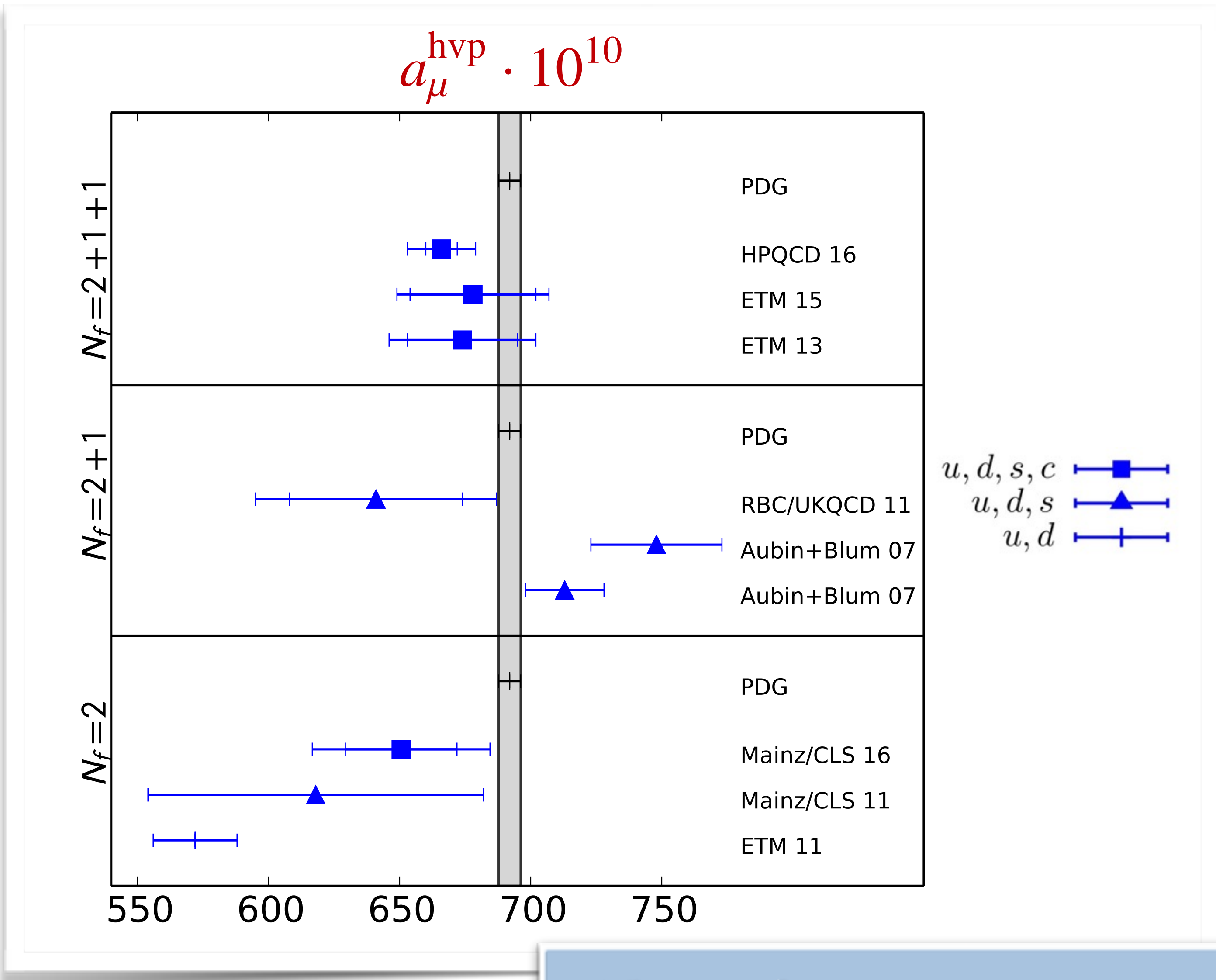
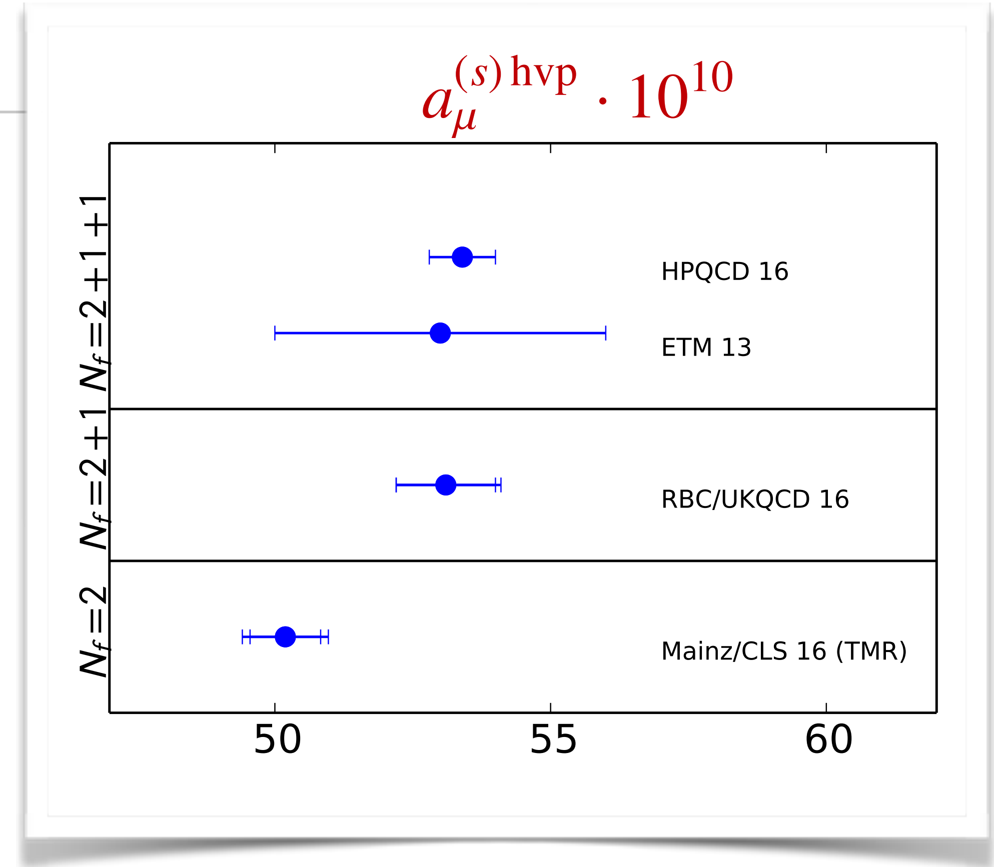


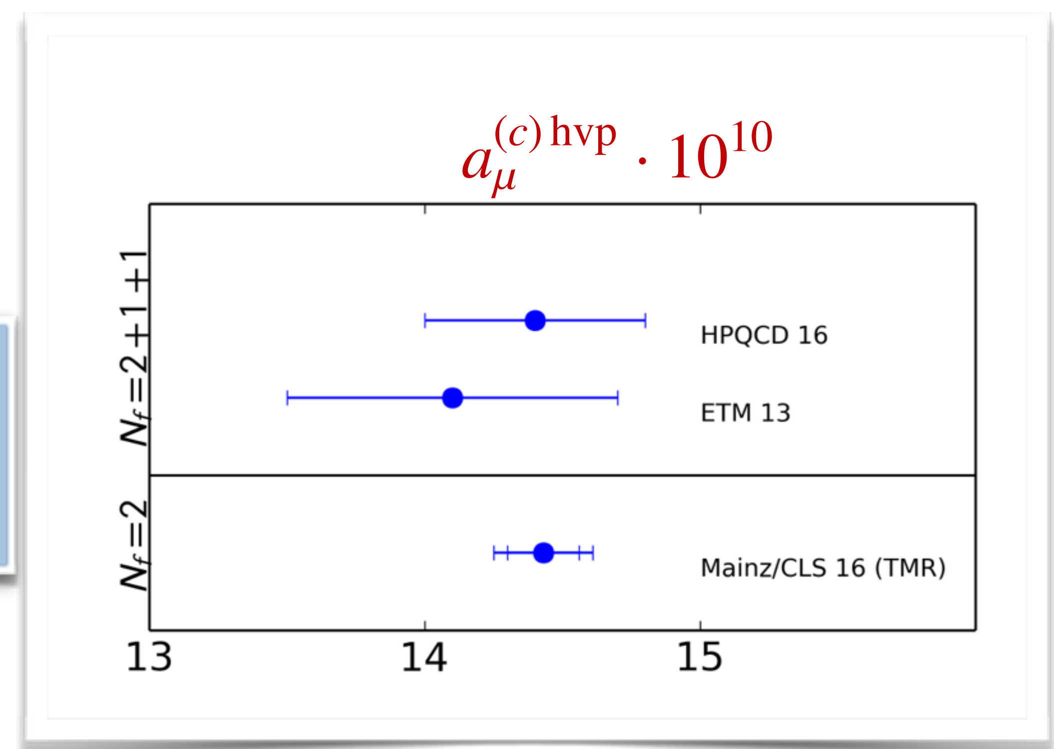
Techniques are there,
need to work on stat. error

Results





Plots from H. Wittig's Lattice 2016 plenary



Summary

- The hadronic contributions to the muon *g*-2 are now a big topic in L(QCD+QED)
- Physical quark mass simulations have allowed for a real breakthrough in reliability
- Tremendous theoretical/algorithmic /computational progress has been made and the prospect of new experimental results keeps the pressure up
- Most concerned about signal-to-noise (long distance) and finite volume effects
- New techniques developed with impact on applications beyond *g*-2
- 1%(10%)-level precision on LO HVP(LbL) are feasible and we will be able to go beyond
- Very exciting times!!!!!!

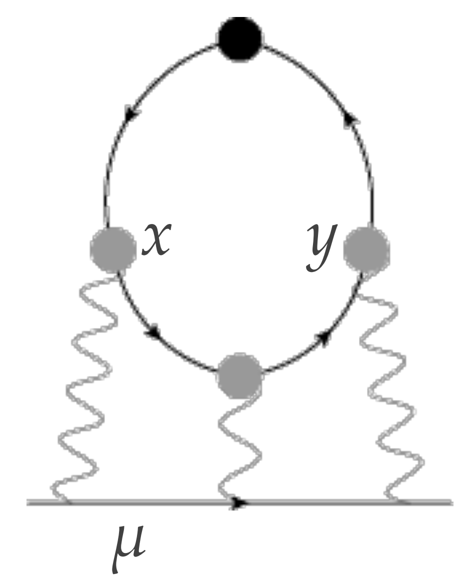
Merci!

LbL via exact photon propagators

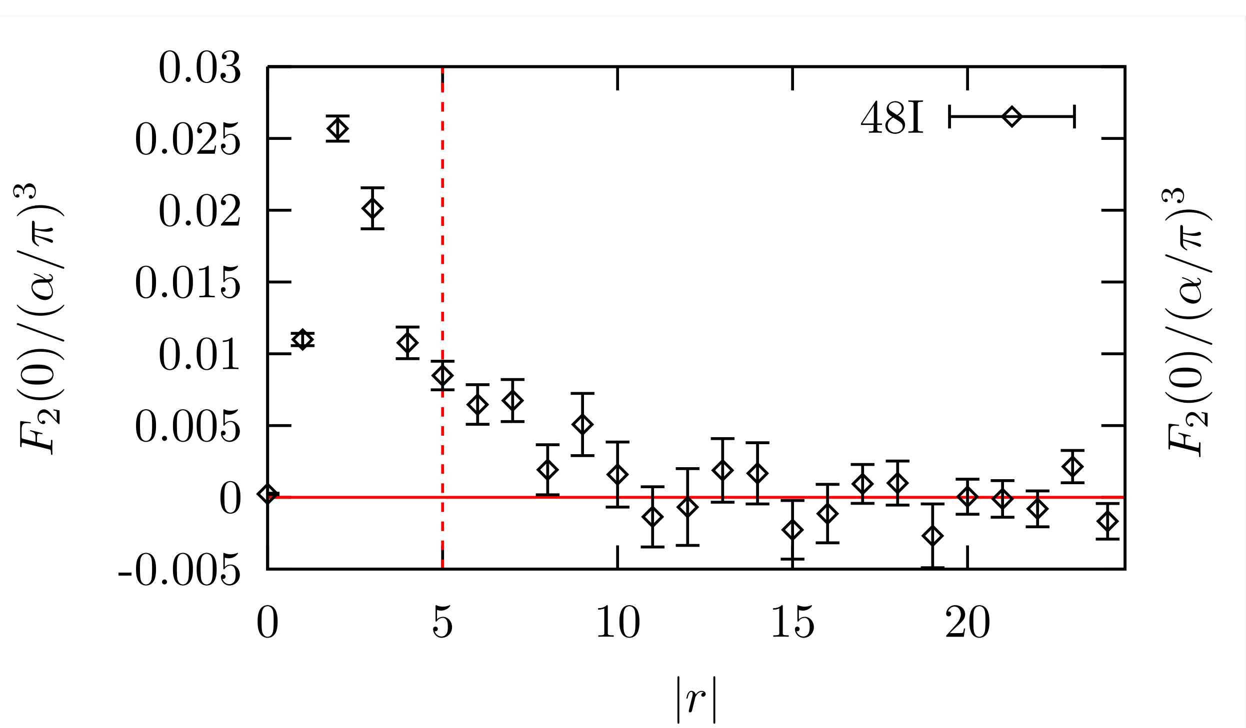
Blum et al. PhysRevD.93.014503 , arXiv:1610.04603

- similar to HVP, moment based approach $(g_\mu - 2)_{cHLbL} \vec{\sigma}_{s's} \propto \int d^3r \left[\vec{r} \times \langle \mu(s') | \vec{J}(\vec{r}) | \mu(s) \rangle \right]$
- perturbative construction including (free) muon propagators
- three Feynman Gauge photon propagators inserted explicitly

$$G_{\mu\nu}(x,y) = \frac{1}{VT} \delta_{\mu\nu} \sum_{k, |\vec{k}| \neq 0} \frac{e^{ik(x-y)}}{\hat{k}^2}$$



- weighted stochastic sampling of x and y position with $r = | x\text{-}y |$



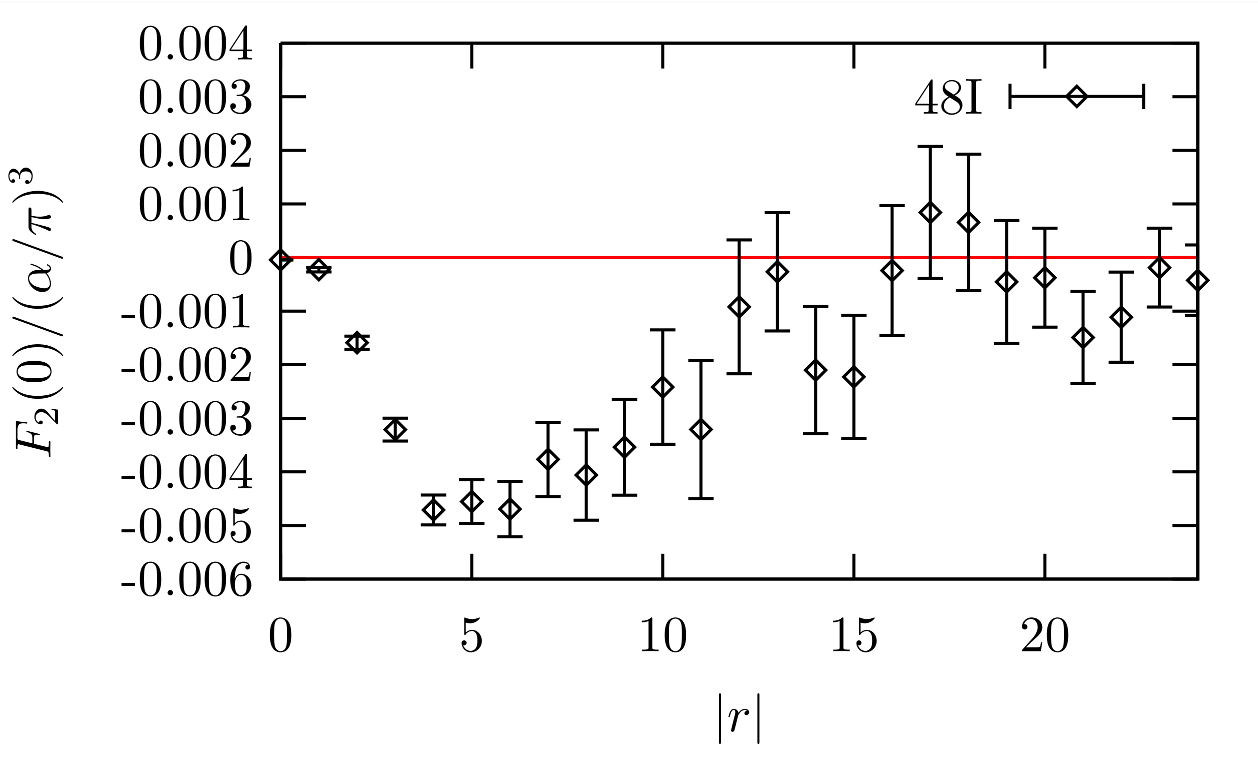
$$a_\mu^{cHLbL} = 11.60(96) \times 10^{-10}$$

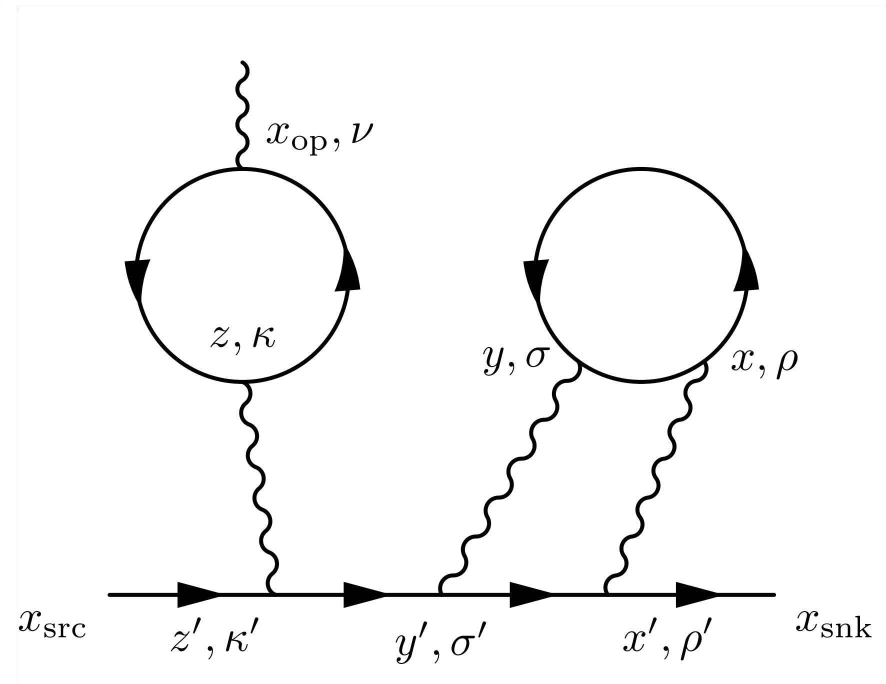
Preliminary result, connected only, further analysis needed

LbL via exact photon propagators

Blum et al. PhysRevD.93.014503 , arXiv:1610.04603

Work on disconnected diagrams under way:





$$a_\mu^{dHLbL} = -6.25(0.80) \times 10^{-10}$$

There is a clear signal for LbL both connected and disconnected contribs, further work on disconnected, finite volume etc. needed but on track…

LbL via exact photon propagators

Blum et al. PhysRevD.93.014503 , arXiv:1610.04603

- First ever physical point results
- preliminary:
  - full set of disco missing
  - finite volume effects to be estimated
  - continuum limit missing

Still a remarkable result: $a_\mu^{HLbL} = -5.35(1.35) \times 10^{-10}$
with finite volume and continuum limit expected to increase the result

to be compared to: e.g. $11.6(3.9)\text{x}10^{-10}$ or $10.5(2.6)\text{x}10^{-10}$,
Jegerlehner, Nyffeler (2009), Prades, de Rafael, Vainstain (2009) and also 1407.4021

Results make it slightly more unlikely that tension can be explained by an error in the LBL calculation