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Dispersion relations for $\gamma\gamma \rightarrow 2\pi$ and $\gamma\gamma^* \rightarrow 2\pi$

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• $\gamma \gamma \rightarrow hadrons$ at e^+e^- colliders:



- 2γ couplings of hadrons $(J^{PC} = 0^{++}, 0^{-+}, 2^{++}, 2^{-+}, 3^{++}, \cdots)$
 - → π^0 , $\eta \rightarrow 2\gamma$: measurement via Primakov [Browman (1974)] was not correct !
 - → σ , $f_0(980)$, $a_0(980) \rightarrow 2\gamma$ Analyticity based extraction: [Mennessier, Z.Phys. C16 (1983) 241]

LBL hadronic contributions to muon g - 2







→ Large N_c approach, next largest: η , σ , f_0 , \cdots poles More general Bern dispersive approach:

Ingredients:

$$\sum_{J} [\gamma \gamma^* \to \pi \pi]_J \times [\gamma^* \gamma^* \to \pi \pi]_J$$

$\gamma\gamma \rightarrow \pi\pi$: analyticity



Independent amplitudes

Amplitude $\gamma^{(*)}(q_1)\gamma^{(*)}(q_2) \rightarrow \pi(p_1)\pi(p_2)$ derived from the matrix element

$$e^{2}W_{\mu\nu}(q_{i},p_{i})=i\int d^{4}x e^{-iq_{1}x}\langle \pi(p_{1})\pi(p_{2})|T\left(j_{\mu}(x)j_{\nu}(0)
ight)|0
angle$$

Ward identities: $\underline{q}_1^{\mu}W_{\mu\nu} = q_2^{\nu}W_{\mu\nu} = 0$.

• Expand on basis $T^n_{\mu\nu}(q_i, p_i)$, $n = 1 \cdots 5$

 $W_{\mu\nu} = A(s, t, q_i^2) T_{\mu\nu}^1 + B(s, t, q_i^2) T_{\mu\nu}^2 + C(s, t, q_i^2) T_{\mu\nu}^3 + \cdots$

Helicity amplitudes: $H_{++}(s, \theta)$, $H_{+-}(s, \theta)$, $H_{+0}(s, \theta)$

Analyticity of scattering amplitudes

 Starting point: Mandelstam analyticity conjecture[PR 112 (1958) 1344]

$$\begin{aligned} A(s, t, u) &= \iint_{D_{st}} ds' dt' \frac{\rho_{st}(s', t')}{(s' - s)(t' - t)} + \\ &\iint_{D_{tu}} dt' du' \frac{\rho_{su}(t', u')}{(t' - t)(u' - u)} + \iint_{D_{us}} du' ds' \frac{\rho_{su}(u', s')}{(u' - u)(s' - s)} \end{aligned}$$

→ Partial-wave amplitudes are analytic functions of *s*

$$\rightarrow \gamma \gamma \rightarrow \pi \pi:$$

$$h'_J(s) = h'_{J,L}(s) + h'_{J,R}(s)$$





Discontinuity across RHC (in elastic region $4m_{\pi}^2 \leq s \leq s_{in}$)

$$\frac{1}{2i} \operatorname{disc}[h'_{J}(s)] = \operatorname{Im}[h'_{J}(s)] = \sigma_{\pi}(s) \underbrace{\left(t'_{J}(s)\right)^{*}}_{\pi\pi} h'_{J}(s)$$

FSI theory [Omnès (1958)]:

$$\Omega_J'(s) = \exp\left[\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'(s'-s)} \Phi_J'(s')\right]$$

with:

$$\begin{split} \varphi'_J(s') &= \delta'_J(s'), \ s' \leqslant s_{in} \ (\text{Fermi-Watson}) \\ &= Phase[h'_J(s')], \ s' > s_{in} \end{split}$$

Application to $2\gamma \rightarrow 2\pi$: [Gourdin,Martin Nuov.Com.17 (1960) 224]

Omnès function removes right-hand cut

$$\operatorname{Im}\left[rac{h_{J}^{\prime}(s)}{\Omega_{J}^{\prime}(s)}
ight]=0,\quad s>4m_{\pi}^{2}$$

General representation:

$$h'_{J}(s) = h'_{J,L}(s) + \Omega'_{J}(s) \left[P_{n-1}(s) + \frac{s^{n}}{\pi} \int_{4m_{\pi}^{2}}^{\infty} ds' \frac{h'_{J,L}(s')\sin(\phi'_{J}(s'))}{(s')^{n}(s'-s)|\Omega'_{J}(s')|} \right]$$

- → Exact, but needs HE information
- → In practice: in finite energy region <u>efficient</u> approximation, <u>few parameters</u>,
- → Fix (partly) from chiral constraints

• Phase ϕ'_J in inelastic region ?

 $\rightarrow \pi \pi_{J=0,I=0}$: main contrib. to inelasticity from $K\overline{K}$

- → Coupled channel extension: Ω'_J is 2 × 2 matrix: must be computed <u>numerically</u> from $\pi\pi$, $K\overline{K}$ 2 × 2 T-matrix
- Prediction for $\phi_0^0 = \text{Phase}[h_0^0]$



Left-hand cut

• Leading contribution at small s: from pion pole in $\gamma \pi^+ \rightarrow \gamma \pi^+$, computed from sQED Lagrangian



- called Born term
- Soft photon theorem [F. Low, PR 110(1958)974]

$$h_J^I(s) \rightarrow h_J^{I,Born}(s) + O(s)$$

 $\Rightarrow P_{n-1}(0) = 0.$



Beyond pion pole: two options

1) Start from Mandelstam based DR's (e.g. family with (t-a)(u-a) = b)[Hoferichter et al. EPJ C71 (2011)1743]. Then, project on PW's:

 $\longrightarrow h_{J,L}(s)$ in terms of $\operatorname{Im} [\gamma \pi \to \gamma \pi]_J$ (but no detailed exp. inputs)

2) Less rigorous (large N_c): resonance contributions to $\gamma \pi \rightarrow \gamma \pi$, from Lagrangian : ρ , ω , a_1 , b_1 , a_2 ...

 \rightarrow Extension to γ^* rather simple



Chiral expansion results and constraints



Historically: $\gamma \gamma \rightarrow \pi^0 \pi^0$:[Bijnens, Cornet NP B296 (1988) 557, Donoghue, Holstein,Lin PR D37 (1988)2423]. One-loop calc. in ChPT: finite, no LEC's

$$H_{++}^{n}(s) = \frac{m_{\pi}^{2} - s}{8\pi^{2}F_{\pi}^{2}} \left(1 + \frac{m_{\pi}^{2}}{s}\log^{2}\frac{\sigma_{\pi}(s) - 1}{\sigma_{\pi}(s) + 1}\right)$$

• Measured at DESY[Crystal Ball, PR D41 (1990) 3324] $\gamma \gamma \rightarrow \pi^0 \pi^0$



Matching DR's and ChPT

• Pion polarisabilities electric (α_i) and magnetic (β_i)

$$\frac{e^2}{2\pi m_{\pi}}H_{++}(s, t = m_{\pi}^2) = s(\alpha_1 - \beta_1) + \frac{s^2}{12}(\alpha_2 - \beta_2) + \cdots$$
$$\frac{-e^2}{2\pi m_{\pi}}H_{+-}(s, t = m_{\pi}^2) = s(\alpha_1 + \beta_1) + \frac{s^2}{12}(\alpha_2 + \beta_2) + \cdots$$

 Polarisabilities in ChPT[Gasser, Ivanov, Sainio NP B728 (2005) 31,B745(2006)84]

π^0	$(\alpha - \beta) [10^{-4} \text{ fm}^3]$	p^4 . p^6 Couplings	
one-loop	-1.0	None	
two-loops	$-(1.9 \pm 0.2)$	<i>C</i> ₂₉ , <i>C</i> ₃₀ , <i>C</i> ₃₁ , <i>C</i> ₃₂ , <i>C</i> ₃₃ , <i>C</i> ₃₄	
π^+			
one-loop	6.0 ± 0.6	$\overline{I_5} - \overline{I_6}$	
two-loops	5.7 ± 1.0	$\cdots + c_6, c_{35}, c_{44}, c_{46}, c_{47}, c_{50}, c_{51}$	
Compass(2014): $(\alpha - \beta)^{\pi^+} = (4.0 \pm 1.2 \pm 1.4) \cdot 10^{-4} \text{ fm}_{14/40}^3$			

Use four constraints from ChPT:

1) For
$$\pi^+$$
: [Gasser et al., NP B745 (2006)84]
 $(\alpha_1 - \beta_1) = [4.7 - 5.7] \, 10^{-4} \, \text{fm}^3$

2) For π^0 : relation between dipole and quadrupole polarisabilities in terms of one p^6 coupling

$$6(\alpha_1 - \beta_1)^{\pi^0} + m_{\pi}^2(\alpha_2 - \beta_2)^{\pi^0} = (6.20 + 10^5 c_{34}^r) 10^{-4} \text{fm}^3$$

 c_{34}^r can be estimated from sum rule: (τ decays inputs) $10^5 c_{34}^r = (1.18 \pm 0.31)$: [Dürr,Kambor PR D61 (2000)] $10^5 c_{34}^r = (1.37 \pm 0.16)$:[Golterman et al.,PR D89 (2014)]

3) For
$$K^+$$
, K^0 : ChPT p^4
 $(\alpha_1 - \beta_1)^{K^0} = 0$, $(\alpha_1 - \beta_1)^{K^+} = \frac{2e^2}{\pi m_K F_K^2} (L_9^r + L_{10}^r)$ 15



16/40







From $\gamma\gamma$ **to** $\gamma\gamma^*(q^2)$



- Goal is to extend the same formalism to $\gamma\gamma^*(q^2) \rightarrow \pi\pi$ (not seemed to have been considered previously)
- Two physically accessible situations:

$$e^-e^+
ightarrow \gamma \pi \pi \qquad (q^2 > 4 m_\pi^2) \ e^- \gamma
ightarrow e^- \pi \pi \qquad (q^2 < 0)$$

- Issues to address
 - 1) "left-hand" cut $(q^2 > 4m_\pi^2)$
 - 2) Form factors



Left-hand cut when $q^2 \neq 0$

Definition of Born term: compute diagrams with sQED and $q^2 \neq 0$



Influence of pion form factor

 $\langle \pi^+(p) | j_\mu(0) | \pi^+(p')
angle = (p + p')_\mu F_\pi^\nu((p - p')^2)$

- \rightarrow First two diagrams mult. by $F_{\pi}^{\nu}(q^2)$
- \rightarrow Gauge invariance: \Rightarrow mult. also third diagram

• Consider J = 0 partial wave projection

$$h_0^{Born}(s, q^2) = \frac{F_{\pi}^{\nu}(q^2)}{s - q^2} \left(\frac{4m_{\pi}^2}{\sigma_{\pi}(s)} \log \frac{1 + \sigma_{\pi}(s)}{1 - \sigma_{\pi}(s)} - 2q^2 \right)$$

with
$$\sigma_{\pi}(s)=\sqrt{1-4m_{\pi}^2/s}$$

 \rightarrow But: if $q^2 > 4m_\pi^2$ pole at $s = q^2$ in the physical region

 \rightarrow Note: $s = q^2$ corresponds to soft photon



→ Omnès dispersive integral:

$$I(s, q^2) = \frac{1}{\pi} \int_{4m_{\pi}^2}^{\infty} \frac{ds' \sin \delta(s') \left(4m_{\pi}^2 L_{\pi}(s') - 2q^2\right)}{(s' - s)(s' - q^2) \left|\Omega(s')\right|}$$

- → well defined: both s, q^2 energy variables, $I(s, q^2)$ defined with $\lim_{\epsilon \to 0} s + i\epsilon$, $q^2 + i\epsilon$
- → <u>Note</u>: Fermi-Watson breaks down[Creutz, Einhorn PR D1 (1970) 2537]

 $\rightarrow \text{Unitarity: Im} \langle \gamma \gamma^* | \pi \pi \rangle \\ = \langle \gamma \gamma^* | \pi \pi \rangle \langle \pi \pi | \pi \pi \rangle + \langle \gamma^* | \pi \pi \rangle \langle \gamma \pi \pi | \pi \pi \rangle$



Further contributions to left-hand cut

Cut :



- \rightarrow Overlaps with positive real axis
- → Problem with Omnès ?



- → Cuts are well separated, Omnès integrals well defined
- → Absence of anomalous threshold. No longer true in case of two virtual photons[Hoferichter et al., arXiv:1309.6877]



Low-energy constraints and Omnès dispersive formula

1) Soft photon limit: $s \rightarrow q^2$

$$H^n_{\lambda\lambda'} = O(s - q^2), \quad H^c_{\lambda\lambda'} = H^{Born}_{\lambda\lambda'} + O(s - q^2)$$

2) Soft pion theorem for $\gamma \gamma^* \to \pi^0 \pi^0$ $A(s, t, q^2) + 2q^2 \left(B(s, t, q^2) - C(s, t, q^2) \right) \Big|_{m_{\pi}=0, t=0, s=0} = 0$ (for any q^2)

⇒ Physical helicity amplitude H_{++} whith $t = m_{\pi}^2$ has an Adler zero i.e. $H_{++}^n(s = 0, t = m_{\pi}^2) = O(m_{\pi}^2)$

Twice subtracted representation (at $\underline{s} = 0$) which generalises the $q^2 = 0$ one

$$\begin{aligned} H_{++}^{I}(s, q^{2}, \theta) &= \\ F_{\pi}^{v}(q^{2}) \, \bar{H}_{++}^{I, Born}(s, q^{2}, \theta) + \sum_{\substack{\rho, \omega \\ \rho, \omega}} F_{V\pi}(q^{2}) \, \bar{H}_{++}^{I, V}(s, q^{2}, \theta) \\ &+ \Omega_{0}^{I}(s) \left\{ s \, F_{\pi}^{v}(q^{2}) \left[\frac{s[J^{I,\pi}(s, q^{2}) - J^{I,\pi}(q^{2}, q^{2})]}{s - q^{2}} - q^{2} \, \hat{J}^{I,\pi}(q^{2}) \right] \\ &+ s \sum_{\rho, \omega} F_{V\pi}(q^{2}) \left[s \, J^{I,V}(s, q^{2}) - q^{2} J^{I,V}(q^{2}, q^{2}) \right] \\ &+ (s - q^{2}) \, b^{I}(q^{2}) \right\} + (J \ge 2) \end{aligned}$$

→ Soft photon condition determines one subtr. function, one remains: $b'(q^2)$

• <u>Remark:</u> $\hat{J}^{I,\pi}$ function involves derivative

$$\hat{J}^{\prime,\pi}(s,q^2) = \frac{1}{\pi} \int_{4m_{\pi}^2}^{\infty} \frac{ds'}{s'-s} \frac{d}{ds'} \left[\frac{\sin \delta_0^{\prime}(s')(4m_{\pi}^2 L_{\pi}(s') - 2q^2)}{|\Omega_0^{\prime}(s')|(s')^2} \right]$$

- → Phase-shift has a cusp at $K\overline{K}$ threshold: principal-value integration diverges (when $s \ge 4m_K^2$)
- Two-channel expression has no problem :

$$\operatorname{Im} \boldsymbol{\Omega}^{-1} = -\boldsymbol{\Omega}^{-1} \times \mathbf{T} \times \begin{pmatrix} \sigma_{\pi}(s')\theta(s'-4m_{\pi}^2) & 0 \\ 0 & \sigma_{K}(s')\theta(s'-4m_{K}^2) \end{pmatrix}$$

and $({f \Omega}^{-1} imes {f T})_{ij}$ has <u>no cusp</u>

Form factors and Subtraction functions

- $F_{\pi\pi}^V$ well known. Similar parametrisations used for $F_{\omega\pi}^V$, $F_{\rho\pi}^V$
- Functions $b'(q^2)$ expected to have cuts $q^2 = 4m_{\pi}^2$, $9m_{\pi}^2$, ..., strong q^2 dep. induced by vector resonances
- Representation w. four parameters (only $\pi^0 \pi^0$ data is available) $b^n(q^2) = b^n(0) \frac{\chi(q^2)}{\chi(0)} + \beta_\rho (GS_\rho(q^2) 1) + \beta_\omega (BW_\omega(q^2) 1)$ $b^c(q^2) = b^c(0) + \beta_\rho (GS_\rho(q^2) 1) + \beta_\omega (BW_\omega(q^2) 1)$

where b^c, bⁿ are linear combinations

$$b^{c} = -(\sqrt{2} b^{0} + b^{2})/\sqrt{6}, \quad b^{n} = -(b^{0} - \sqrt{2} b^{2})/\sqrt{3}$$

• Function $\chi(q^2)$ from $O(p^4) \pi^0 \pi^0$ amplitude at s = 0

$$\chi(q^2) = \frac{-2m_{\pi}^2(\bar{G}_{\pi}(q^2) - \bar{J}_{\pi}(q^2))}{F_{\pi}^2 q^2}$$

Note:

$$\chi(q^2)$$
 is $\underline{O(m_\pi^2)}$ if $q^2 \neq 0$
 $\chi(0) = -1/(96\pi^2 F_\pi^2)$ is $\underline{O(1)}$.

• $q^2 = 0$: $b^n(0)$, $b^c(0)$ constrained from polarisabilities

$$\begin{aligned} (\alpha_1 - \beta_1)^{\pi^0} &= \frac{2\alpha}{m_{\pi}} \left[b^n(0) - 4m_{\pi}^2 \widetilde{C}_{\rho^0} \widetilde{BW}_{\rho}(m_{\pi}^2) - \frac{4m_{\pi}^2 \widetilde{C}_{\omega}}{m_{\omega}^2 - m_{\pi}^2} \right] \\ (\alpha_1 - \beta_1)^{\pi^+} &= \frac{2\alpha}{m_{\pi}} \left[b^c(0) - 4m_{\pi}^2 \widetilde{C}_{\rho^+} \widetilde{BW}_{\rho}(m_{\pi}^2) \right] \end{aligned}$$



• $\underline{q^2 \neq 0}$: Data on $e^+e^- \rightarrow \gamma^* \rightarrow \pi^0 \pi^0 \gamma$ [Akhmetshin et al.[CMD2], Phys.Lett.B580(2004)119,Achasov et al., [SND], Phys.Lett.B537 (2002) 201]

Two parameters fit:

β _ρ	β _w	χ^2/N_{dof}	ref.
0.14 ± 0.12	$(-0.39\pm0.12)10^{-1}$	20.2/27	SND (2002)
-0.13 ± 0.15	$(-0.31\pm0.15)10^{-1}$	15.0/21	CMD-2 (2003)
0.05 ± 0.09	$(-0.37\pm0.09)10^{-1}$	38.1/50	Combined

• $e^+e^- o \gamma^* o \pi^0\pi^0\gamma$ data is well reproduced (but not very precise)





 $\gamma \pi \pi$ contribution to a_{μ} via HVP

Generic expression

[Lautrup, de Rafael (1974)]

$$a_{\mu}^{\pi\pi\gamma}[q_{max}] = \frac{1}{4\pi^3} \int_{4m_{\pi}^2}^{q_{max}^2} dq^2 \, K_{\mu}(q^2) \, \sigma_{e^+e^- \to \pi\pi\gamma}(q^2)$$

 \rightarrow In terms of helicity amplitudes

$$\sigma(q^{2}) = \frac{\alpha^{3}}{12(q^{2})^{3}} \int_{4m_{\pi}^{2}}^{q^{2}} ds(q^{2}-s)\sigma_{\pi}(s) \int_{-1}^{1} dz \sum |H_{\lambda\lambda'}(s,q^{2},\theta)|^{2}$$

 \rightarrow For <u>charged</u> pions

$$\begin{aligned} |H_{\lambda\lambda'}^{c}|^{2} &= \underbrace{|H_{\lambda\lambda'}^{Born}|^{2}}_{\sigma = \sigma^{sQED}} + \underbrace{2\text{Re}[H_{\lambda\lambda'}^{Born}\hat{H}_{\lambda\lambda'}^{*}]}_{\sigma Born} + \underbrace{|\hat{H}_{\lambda\lambda'}|^{2}}_{\sigma \sigma} \end{aligned}$$

33

• σ^{sQED} : made finite by adding rad. corr. of $\gamma^* \pi^+ \pi^-$ vertex $\sigma^{sQED} = \frac{\pi \alpha^3}{3q^2} \sigma_{\pi}^3(q^2) |F_{\pi}^{\nu}(q^2)|^2 \times \frac{\alpha}{\pi} \eta(s)$ [Jegerlehner, Nyffeler PRep. 477(2009)1]

Numerical results (q^{max} = 0.95 GeV):

channel	cross-section	a _µ
$\gamma\pi^+\pi^-$	σ ^{sQED}	41.9×10^{-11}
$\gamma\pi^+\pi^-$	σ ^{Born}	$(1.31\pm0.30) imes10^{-11}$
$\gamma\pi^+\pi^-$	σ	$(0.16\pm0.05) imes10^{-11}$
$\gamma \pi^0 \pi^0$	$\sigma^{\gamma\pi^0\pi^0}$	$(0.33\pm0.05) imes10^{-11}$

→ For comparison: [Davier et al. (2010)] $a_{\mu}^{\pi\pi(\gamma)} = (5078.0 \pm 12.2 \pm 25.0 \pm 5.6) \times 10^{-11}$

Conclusions

$\gamma\gamma \to \pi\pi$:

- 1) In region $E_{\pi\pi} \leq 0.8$ GeV: parameter-free representation, once matching w. ChPT near s = 0 (+ phase with a dip in Omnès funct.)
- 2) In larger region $E_{\pi\pi} \leq 1.3$ GeV, with CC unitarity in J = 0, 6 params repres. fits the data.
- 3) Larger angular coverage needed !

 $\gamma\gamma^* \to \pi\pi$:

- 1) Extension of one-channel formalism presented: involves two subtractions functions of $q^2\,$
- 2) Available data $e^+e^- \rightarrow \gamma \pi^0 \pi^0$ (Novosibirsk) reproduced. Data from KLOE at $q^2 \simeq m_{\Phi}^2$ requires CC extension.
- 3) Application: $\gamma \pi \pi$ contrib. in HVP to muon g 2: beyond sQED, very small.



Extra slides





- Pion form factor is well studied experimentally[Babar PR D86(2012)032013, KLOE PL B670(2009)285...]
 - → Parametris. with q² analyticity and rel. to ππ J = 1 phase-shifts[Colangelo, NP B (proc.sup.) 162(2006)256]

 $\begin{array}{ll} \underline{\omega\pi} \text{ form factor: data also exists} \\ 1) & q^2 > (m_{\omega} + m_{\pi})^2 \mbox{ (From } e^+e^- \to \omega\pi) \\ 2) & q^2 < (m_{\omega} - m_{\pi})^2 \mbox{ (From } \omega \to l^+l^-\pi) \end{array}$

Use simple representation:

$$\begin{aligned} \mathcal{F}_{\omega\pi}(q^2) &= \\ \frac{1}{1+\beta'} \Big[GS_{\rho}(q^2) \Big(1 + \frac{\delta \frac{q^2}{m_{\omega}^2}}{m_{\omega}^2} BW_{\omega}(q^2) \Big) + \beta' GS_{\rho(1450)}(q^2) \Big] \end{aligned}$$

Data in range $q^2 > (m_{\omega} + m_{\pi})^2$ well reproduced

Also data in range $\sqrt{q^2} < 0.6 \text{ GeV}$

 Some data points problematic





form factor: no data exists in this case. Plausible guess: assume three I = 0 resonances dominate

$$F_{\rho\pi}(q^2) = \alpha_{\omega} BW_{\omega}(q^2) + \alpha_{\varphi} BW_{\varphi}(q^2) + \alpha_{\omega'} BW_{\omega'}(q^2)$$

with $(\alpha_{\omega}+\alpha_{\varphi}+\alpha_{\omega'}=1).$ Use relations

$$\alpha_{V} = \frac{F_{V}g_{V\rho\pi}}{2m_{V}C_{\rho}}, \quad V = \omega, \ \varphi$$

and phenomenological determinations of $g_{V\rho\pi}$

→ Fortunately: this form factor is much less important numerically than $F_{\omega\pi}$

Comparison of chiral and dispersive H_{++}^n

• $\frac{q^2 < 0}{\text{Adler zero is present}}$



• $q^2 > 0.3$ (GeV²) Adler zero disappears