Generalized Parton Distributions and their covariant extension

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Outline

1. Introduction to Generalized Parton Distributions
   - Definition and properties
   - Experimental access

2. Modeling Generalized Parton Distributions
   - Overlap of Light-cone wave functions
   - Double Distributions
   - Inversion of Incomplete Radon Transform
   - Results

3. Conclusion
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Definition of GPDs

- Quark GPD (twist-2, spin-0 hadron): \((\text{Müller et al., 1994; Radyushkin, 1996; Ji, 1997})\)

\[
H^q(x, \xi, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{i \times P^+ z^-} \left\langle \begin{array}{c} P + \frac{\Delta}{2} \\ \bar{q}(-z) \gamma^+ q(z) \end{array} \middle| P - \frac{\Delta}{2} \right\rangle \bigg|_{z^+ = 0, z_\perp = 0}. 
\]

with:

\[
t = \Delta^2, \quad \xi = -\frac{\Delta^+}{2P^+}.
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- **Impact parameter space GPD (at \(\xi = 0\)):** (Burkardt, 2000)

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  - DGLAP: \(|x| > |\xi|\).
- Link to PDFs and Form Factors:
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  \]
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  H_q(x, 0, 0) = \theta(x) q(x) - \theta(-x) \bar{q}(-x).
  \]
- Polynomiality:
  \[
  \int_{-1}^{1} dx x^m H_q(x, \xi, t) = \text{Polynomial in } \xi.
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- Positivity (in DGLAP):
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  |H_q(x, \xi, t)| \leq \sqrt{q(x - \xi) q(x + \xi)}.
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  - Cauchy-Schwarz theorem in Hilbert space.
Accessing GPDs

- Exclusive processes:

- DVCS
- TCS
- DVMP

Observables are convolutions of:

- A hard-scattering kernel, calculated with perturbative QCD (short distance interactions).
- A soft part, i.e. the GPD, with long distance interactions encoded (non-perturbative QCD).
Accessing GPDs

- **Exclusive processes:**

\[
\begin{align*}
\ell & \rightarrow \ell \\
\gamma & \rightarrow GPD \\
\gamma^* & \rightarrow GPD \\
\rho^0, \phi, \pi^0, \ldots & \rightarrow GPD
\end{align*}
\]

- **Compton Form Factors:** (Belitsky et al., 2002)

\[
\mathcal{F}(\xi, t, Q^2) = \int_{-1}^{1} dx \ C \left( x, \xi, \alpha_s(\mu_F), \frac{Q}{\mu_F} \right) F(x, \xi, t, \mu_F). \tag{7}
\]
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Overlap of Light-cone wave functions

• A given hadronic state is decomposed in a Fock basis: (Brodsky et al., 1981)

\[ |H; P, \lambda\rangle = \sum_{N,\beta} \int [dx]_{N} [d^{2}k_{\perp}]_{N} \Psi_{N,\beta}^{\lambda}(x_{1}, k_{\perp 1}, \ldots) |N, \beta; k_{1}, \ldots, k_{N}\rangle, \quad (10) \]

where the \( \Psi_{N,\beta}^{\lambda} \) are the Light-cone wave-functions (LCWF).
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- For example, for the pion:

\[ |\pi^+\rangle = \psi_{u\bar{d}}^{\pi} |u\bar{d}\rangle + \psi_{u\bar{d}g}^{\pi} |u\bar{d}g\rangle + \ldots \tag{11} \]
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- GPD as an overlap of LCWFs: \( (\text{Diehl et al., 2001; Diehl, 2003}) \)

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H^q(x, \xi, t) = \sum_{N, \beta} \sqrt{1 - \xi^2}^{-N} \sqrt{1 + \xi^2}^{-N} \sum_a \delta_{a,q} \times \int [dx]_N [d^2 \vec{k}_\perp]_N \delta(x - \bar{x}_a) \psi^*_{N, \beta}(\hat{x}_1', \hat{k}_{\perp 1}', \ldots) \psi_{N, \beta}(\hat{x}_1, \hat{k}_{\perp 1}, \ldots) , \tag{12}
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in the DGLAP region \( \xi < x < 1 \) (pion case).
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  - Polynomiality not manifest...
Double Distributions (DDs)

- DD representation of GPDs:

\[ H(x, \xi, t) \propto \int_{\Omega} d\beta d\alpha f(\beta, \alpha, t) \delta(x - \beta - \alpha \xi). \]  

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\propto \int_{\Omega} d\beta d\alpha (\beta + \xi \alpha)^m f(\beta, \alpha, t) .
\]  

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  - Need ERBL to complete **polynomiality**.
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**Problem**

Find $f(\beta, \alpha)$ on square $\{|\alpha| + |\beta| \leq 1\}$ such that

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- Quark GPD: $H(x, \xi) = 0$ for $-1 < x < -|\xi| = \Rightarrow f(\beta, \alpha) = 0$ for $\beta < 0$.
- Domains $\beta < 0$ and $\beta > 0$ are uncorrelated in the DGLAP region.
- Divide and conquer:
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  - Lesser complexity: $O(N_p + N_p) \ll O((N + N) p)$. 
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GPDs and covariant extension
Pheniics Fest 30/05/17
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Theoretical “L-curve”: curve parameterized by the regularization factor.

(fig. taken from Ref. \( \text{(Hansen, 2007)} \))
Some examples (Dyson-Schwinger model)

Figure: Extension of GPDs for the pion DSE model of Refs. (Mezrag, 2015; Mezrag et al., 2016). Left: $t = 0$ GeV$^2$. Right: $t = 1$ GeV$^2$. Comparison to the analytical result.
Some examples (Spectator model)

**Figure:** Extension of GPD $E$ for the nucleon model of Ref. (Hwang and Mueller, 2008). Comparison to the analytical result of the paper.
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Modeling GPDs

Conclusion

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  - Any questions?


