Ecole de Gif 2017

Relativité générale et ondes gravitationnelles

GENERAL RELATIVITY AND ALTERNATIVE THEORIES

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1. Metric theories of gravity
2. Polarization modes in metric theories
3. General relativity
4. Scalar-tensor theories
5. Massive gravity and bimetric theories
6. Einstein-Cartan theory

METRIC THEORIES OF GRAVITY

Based on 2 fundamental principles

- <u>principle of special relativity</u> (Einstein, Lorentz, Poincaré 1905)
- <u>(weak) equivalence principle</u> or universality of free fall

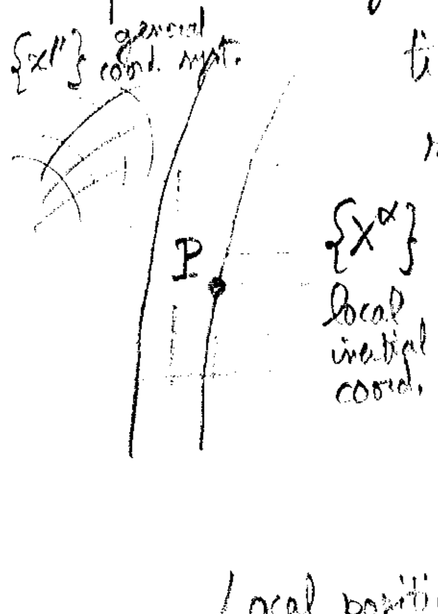
theories	Metric theories	
principles	ppe of relat	ppe of equivalence (WEP)
Experiments	Fizeau (1851) Michelson-Morley (1887) $\frac{\delta c}{c} \lesssim 10^{-8}$ (Brillet-Hall 1979) Hughes-Drever ⋮	Eötvös (1898) Dicke et al, Braginsky $\left\lvert \left(\frac{m_i}{m_g}\right)_a - \left(\frac{m_i}{m_g}\right)_b \right\rvert \lesssim 10^{-12}$ Microscope (10^{-15} ?) Lunar laser ranging (10^{-13})

These two principles can be combined into the <u>Einstein equivalence principle</u> EEP (1911)

1. WEP is valid hence all test bodies (uncharged, small extension w.r.t. inhomogeneities of the grav. field) have the same acceleration in a gravitational field, independently of their internal structure and composition (and mass);
2. In the freely falling frame of test bodies, called <u>local inertial frames</u>, the laws of special relativity are valid.

In particular

<u>Local Lorentz invariance</u> (LLI): the outputs of local experiments (performed in the freely falling inertial frame) are independent of the velocity of that frame. This implies that a non-gravitational interaction A must be coupled to a second-rank tensor field $\psi^{(A)}_{\mu\nu}(x)$ reducing in the local inertial coord. to



$$\psi^{(A)}_{\alpha\beta}(X) = \phi^{(A)}(X)\,\eta_{\alpha\beta}$$

$$\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1) \qquad \text{Minkowski metric} \quad \alpha,\beta = 0,1,2,3$$

<u>Local position invariance</u> (LPI): outputs of experiments are independent of the space-time location (event $P = (x^\mu) = (X^\alpha)$). Hence*

$$\phi^{(A)}(X) = c^A\,\phi(X)$$

and c^A can be set to one by rescaling the coupling constant of the interaction A. Hence we conclude that all non-gravitational interactions A must be <u>universally coupled</u> to a single tensor field

$$\boxed{g_{\mu\nu} = \phi^{-1}\psi_{\mu\nu}} \quad \text{in arbitrary coordinates } \{x\},$$

which reduces to $g_{\alpha\beta} = \eta_{\alpha\beta}$ at P in loc. inertial coord. $\{X\}$.

Postulates of EEP are equivalent to those of <u>metric theories of gravity</u>: (1) Space-time is endowed with a metric $g_{\mu\nu}$

$$\boxed{ds^2 = g_{\mu\nu}(x)\,dx^\mu dx^\nu} \qquad \mu,\nu = 0,1,2,3$$

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* Measured physical quantities are dimensionless ratios and $\phi(X)$ will cancel out. For instance the length of a rod is measured by comparing to a standard rod whose length is defined to be one meter.

(2) The worldlines of test bodies are <u>geodesics</u> of that metric:

$$\boxed{a^\mu = u^\nu \nabla_\nu u^\mu = 0}$$

$$\begin{cases} u^\mu = \dfrac{dx^\mu}{d\tau} & \text{4-velocity} \\ a^\mu & \text{4-acceleration} \end{cases}$$

$d\tau^2 = -\frac{1}{c^2} ds^2$ proper time of test body

covariant derivative $\nabla_\nu u^\mu = \partial_\nu u^\mu + \Gamma^\mu_{\nu\rho} u^\rho$

$$\underbrace{\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\lambda}\left(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho}\right)}_{\text{Christoffel symbol}}$$

u^μ, a^μ are tensors $\begin{bmatrix}1\\0\end{bmatrix}$, $g_{\mu\nu}$ is a tensor $\begin{bmatrix}0\\2\end{bmatrix}$, ds^2 is a scalar $\begin{bmatrix}0\\0\end{bmatrix}$,

$\nabla_\nu u^\mu$ is a tensor $\begin{bmatrix}1\\1\end{bmatrix}$, but $\Gamma^\mu_{\nu\rho}$ is <u>not</u> a tensor.*

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* A tensor $\begin{bmatrix}p\\q\end{bmatrix}$ is defined by its components in an arbitrary coord. system $\{x\}$

$$T^{\overbrace{\mu\nu\cdots}^{p\text{ indices}}}_{\underbrace{\rho\sigma\cdots}_{q\text{ indices}}}(x)$$

In an arbitrary change of coord. $\{x\} \longrightarrow \{x'\}$

$$T'^{\mu\nu\cdots}_{\rho\sigma\cdots}(x') = \frac{\partial x'^\mu}{\partial x^\eta} \frac{\partial x'^\nu}{\partial x^\varepsilon} \cdots \frac{\partial x^\lambda}{\partial x'^\rho} \frac{\partial x^\tau}{\partial x'^\sigma} \, T^{\eta\varepsilon\cdots}_{\lambda\tau\cdots}(x)$$

The covariant derivative of tensor $\begin{bmatrix}p\\q\end{bmatrix} \longrightarrow$ tensor $\begin{bmatrix}p\\q+1\end{bmatrix}$

$$\nabla_\lambda T^{\mu\nu\cdots}_{\rho\sigma\cdots} = \partial_\lambda T^{\mu\nu\cdots}_{\rho\sigma\cdots} + \Gamma^\mu_{\lambda\varepsilon} T^{\varepsilon\nu\cdots}_{\rho\sigma\cdots} + \Gamma^\nu_{\lambda\varepsilon} T^{\mu\varepsilon\cdots}_{\rho\sigma\cdots}$$
$$- \Gamma^\varepsilon_{\lambda\rho} T^{\mu\nu\cdots}_{\varepsilon\sigma\cdots} - \Gamma^\varepsilon_{\lambda\sigma} T^{\mu\nu\cdots}_{\rho\varepsilon\cdots}$$

All physical (geometrical) objects in metric theories are tensors.

(3) In local inertial frames $\{X^\alpha\}$ the metric reduces to the Minkowski metric and its derivatives vanish

$$\boxed{g_{\alpha\beta} = \eta_{\alpha\beta} \quad \text{and} \quad \frac{\partial g_{\alpha\beta}}{\partial X^\gamma} = 0 \quad \text{at event P in loc. in. coord. } \{X\}}$$

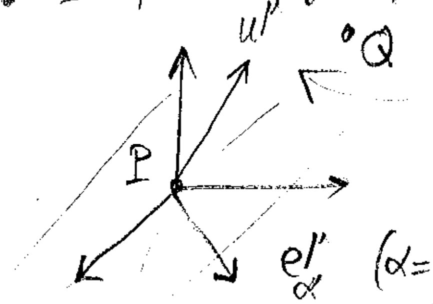
This implies $\boxed{\nabla_\rho g_{\mu\nu} = 0}$ (Ricci theorem) in any coordinate system $\{x\}$.

Intrinsic curvature of space-time is defined by the <u>Riemann curvature tensor</u> $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$ $R_{\mu\nu\rho\sigma}$

- <u>Non-commutation of covariant derivatives</u> (Ricci identity)*

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\lambda = R^\lambda{}_{\bullet\varepsilon\mu\nu} V^\varepsilon$$

- <u>Impossibility of defining global inertial coordinates</u>



geodesics from P to Q with unit vector u^μ ($g_{\mu\nu} u^\mu u^\nu |_P = \pm 1$)

e^μ_α ($\alpha = 0,1,2,3$)
orthonormal tetrad at P

$$g_{\mu\nu} e^\mu_\alpha e^\nu_\beta \big|_P = \eta_{\alpha\beta}$$

s = s.t. interval of position of Q w.r.t. P along geodesics

<u>Riemann normal coordinates</u> $\{X^\alpha\}$ defined by

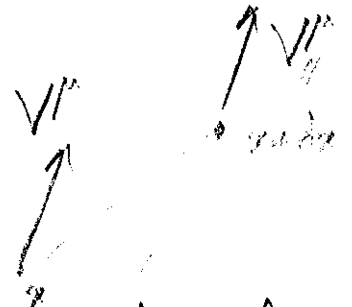
$$s\, u^\mu = X^\alpha e^\mu_\alpha$$

$$\boxed{g_{\alpha\beta} = \eta_{\alpha\beta} + \frac{1}{3} R_{\alpha\beta\gamma\delta} X^\gamma X^\delta + O(X^3)}$$

* We have

$$R^\lambda{}_{\bullet\rho\sigma\nu} = \partial_\sigma \Gamma^\lambda_{\rho\nu} - \partial_\nu \Gamma^\lambda_{\rho\sigma} + \Gamma^\mu_{\rho\nu} \Gamma^\lambda_{\mu\sigma} - \Gamma^\mu_{\rho\sigma} \Gamma^\lambda_{\mu\nu}$$

- Ambiguity in the parallel transport along a closed contour



eq. of parallel transport

$$\boxed{dx^\nu \nabla_\nu V^\mu = 0}$$

satisfied by vector V^μ $\left(V^\mu_{\parallel} = V^\mu\right)$

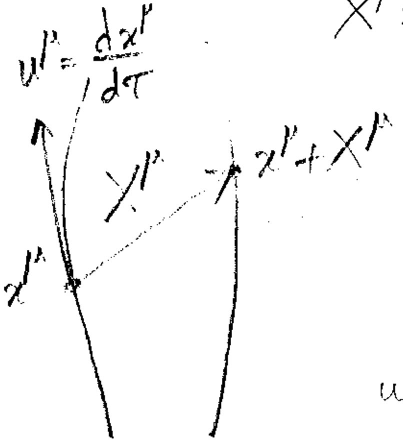
Along the closed (infinitesimal) contour C after one round trip

$$\boxed{\Delta V^\mu = V^\mu_{\parallel} - V^\mu = -\frac{1}{2} R^\mu_{\cdot\nu\rho\sigma} V^\nu \oint_C x^\rho dx^\sigma}$$

where $\oint_C x^\rho dx^\sigma = \int_{\substack{\text{surface with}\\ \text{boundary } C}} dx^\rho \wedge dx^\sigma$ = surface area sustained by C (antisymmetric in $\rho\sigma$)

- Equation of geodesic deviation

X^μ = small separation between two geodesics



$$\boxed{\frac{D^2 X^\mu}{d\tau^2} = - R^\mu_{\cdot\nu\rho\sigma} u^\nu X^\rho u^\sigma}$$

where $\frac{D}{d\tau} = u^\nu \nabla_\nu$

- Bianchi and Einstein identities

$$\boxed{\nabla_\lambda R^\mu_{\cdot\nu\rho\sigma} + \nabla_\rho R^\mu_{\cdot\nu\sigma\lambda} + \nabla_\sigma R^\mu_{\cdot\nu\lambda\rho} \equiv 0}$$

By contracting two times

$$\boxed{\nabla_\nu G^{\mu\nu} \equiv 0}$$

with $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ Einstein tensor

$R^{\mu\nu} = R^{\lambda\mu\nu}_{\cdot\cdot\lambda\cdot}$ Ricci tensor

$R = g^{\mu\nu} R_{\mu\nu}$ scalar curvature

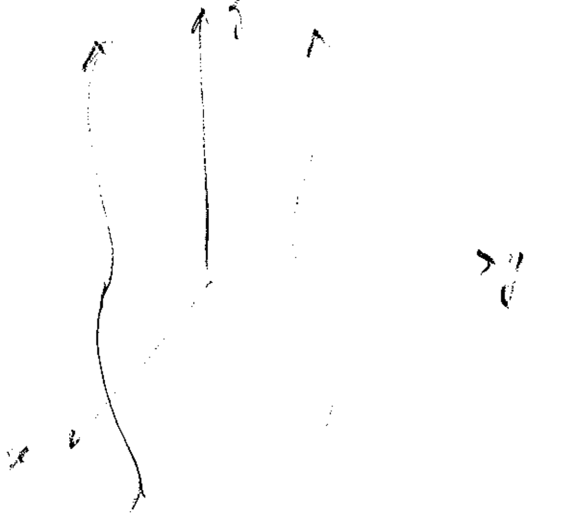
Plane gravitational waves in metric theories

We consider a wave-like small perturbation of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{with } |h_{\mu\nu}| \ll 1$$

We restrict attention to metric theories admitting wave solutions propagating at the speed of light $c = 1$.* Far from their sources the waves are almost planar. We choose the axes of the coord. system such that the wave propagate in the z direction hence $\Box h_{\mu\nu} = 0$ gives

$$h_{\mu\nu} = h_{\mu\nu}(t-z)$$



From that let us prove that the components of the Riemann tensor also propagate with the speed of light

$$\boxed{R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}(t-z)}$$

This follows from the Bianchi identities at order $\mathcal{O}(h^2)$

$$\partial_\lambda R^{\mu}_{\cdot\nu\rho\sigma} + \partial_\rho R^{\mu}_{\cdot\nu\sigma\lambda} + \partial_\sigma R^{\mu}_{\cdot\nu\lambda\rho} = \mathcal{O}(h^2)$$

Contracting with δ^ρ_μ this gives

$$\partial_\rho R^{\rho}_{\cdot\nu\sigma\lambda} = \partial_\sigma R_{\nu\lambda} - \partial_\lambda R_{\nu\sigma} + \mathcal{O}(h^2)$$

* Perhaps only in certain gauges, like in GR. Some metric theories (probably with prior geometric elements, such as a fixed Minkowski metric) have $c_{GW} \neq c_{EM}$. We exclude these cases.

Hence, taking now the divergence ∂^λ of Bianchi

$$\Box R_{\mu\nu\rho\sigma} = \partial_\rho\left(\partial_\mu R_{\nu\sigma} - \partial_\nu R_{\mu\sigma}\right) - \partial_\sigma\left(\partial_\mu R_{\nu\rho} - \partial_\nu R_{\mu\rho}\right) + O(h^2)$$

Further expressing Ricci in terms of $h_{\mu\nu}$

$$R_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial_\mu\partial_\rho h^\rho_\nu + \partial_\nu\partial_\rho h^\rho_\mu\right) + O(h^2)$$

this gives (exercice)

$$\boxed{\Box R_{\mu\nu\rho\sigma} = -\frac{1}{2}\partial_\mu\left(\partial_\rho \Box h_{\nu\sigma} - \partial_\sigma \Box h_{\nu\rho}\right) + \frac{1}{2}\partial_\nu\left(\partial_\rho \Box h_{\mu\sigma} - \partial_\sigma \Box h_{\mu\rho}\right) + O(h^2)}$$

Hence since $\Box h_{\mu\nu} = 0$ we have also $\Box R_{\mu\nu\rho\sigma} = 0$.

We look for the number of independent components of $R_{\mu\nu\rho\sigma}$ (t-3)

Using again the (linearized) Bianchi identity together with the known symmetries of Riemann

$$\left.\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma} = R_{\nu\mu\sigma\rho} \\ R_{\mu\nu\rho\sigma} &= R_{\rho\sigma\mu\nu} \\ R^\lambda_{\ \sigma\mu\nu} + R^\lambda_{\ \mu\nu\sigma} &+ R^\lambda_{\ \nu\sigma\mu} = 0 \end{aligned}\right\}$$

give 20 independent components of Riemann in 4 dimensions

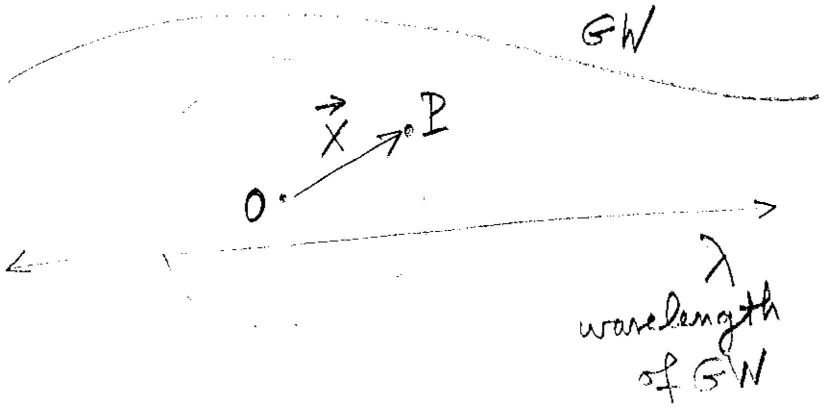
this yields the constraints, for any pair $\mu\nu$ (exercice)

$$R_{\mu\nu xy} = 0 \qquad R_{\mu\nu ox} = R_{\mu\nu xz} \qquad R_{\mu\nu oy} = R_{\mu\nu yz}$$

Thus $R_{0i0j}(t-z)$ where $i,j = x,y,z$ represent 6 independent components for the most general wave.

<u>Gravitational waves</u> (propagating at the speed of light $c=1$) <u>in metric theories of gravity have 6 independent polarizations.</u>

We study the deformation of a sphere of particles freely falling in the field of the GW for each of the polarization modes.



Work out the equation of geodesic deviation between "central" geodesic of O and the geodesic of a point P on the sphere, at first order in

$$\boxed{\frac{|\vec{X}|}{\lambda} \ll 1}$$

$$\boxed{\frac{D^2 X^\mu}{d\tau^2} = - R^\mu_{\ \cdot \nu\rho\sigma} u^\nu X^\rho u^\sigma}$$

where we can always choose X^μ such that $u_\mu X^\mu = 0$, which follows from the fact that $u_\mu \frac{D^2 X^\mu}{d\tau^2} = 0$ (exercice).

The 4-velocity u^μ is the velocity of the central geodesic followed by O and since u^μ is always multiplied by terms $\mathcal{O}(h)$ we can choose $u^\mu = (1, \vec{0})$ hence

$$X^\mu = (0, X^i) \quad \text{modulo } \mathcal{O}(h) \text{ terms}$$

We adopt a Fermi normal coordinate system $(T, \vec{X})$ for which the metric deviates from the flat metric by small terms in the spatial distance

$$g_{\alpha\beta} = \eta_{\alpha\beta} + f_{\alpha\beta ij}(T) X^i X^j + \mathcal{O}(|\vec{X}|^3)$$

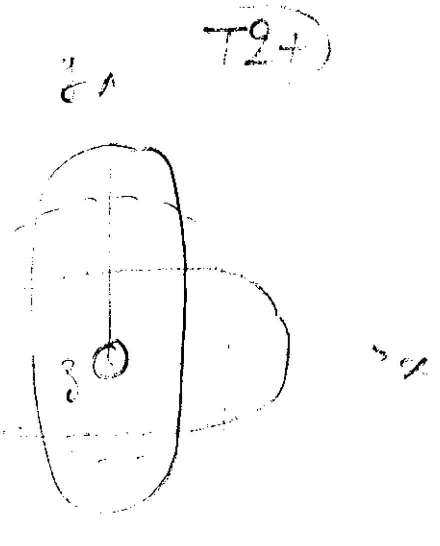
In Fermi coordinates $\Gamma^\gamma_{\alpha\beta} = \mathcal{O}(\vec{X})$, $\partial_i \Gamma^\gamma_{\alpha\beta} = \mathcal{O}(?)$ but the time derivative of $\Gamma^\gamma_{\alpha\beta}$ is small: $\partial_T \Gamma^\gamma_{\alpha\beta} = \mathcal{O}(\vec{X})$. Thanks to that we see that

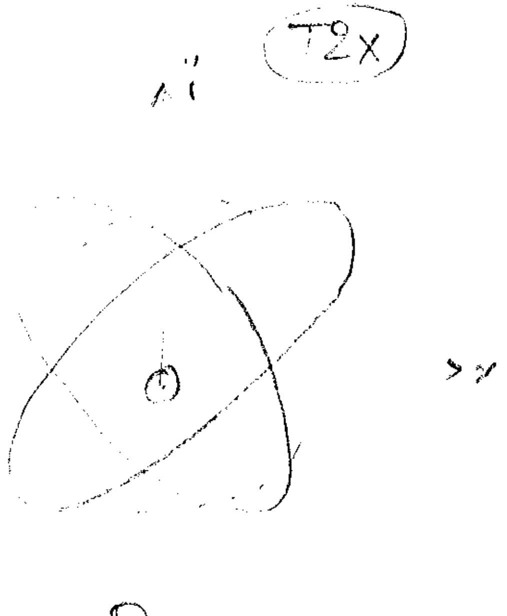
$$\frac{D^2 X^i}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dX^i}{d\tau} + \Gamma^i_{0j} X^j\right) + \Gamma^i_{0j}\left(\frac{dX^j}{d\tau} + \Gamma^j_{0k} X^k\right) = \frac{d^2 X^i}{d\tau^2} + \mathcal{O}(\vec{X}^2)$$

and $\tau \simeq T$ for non-relativistic motion. Hence, in Fermi coordinates,

$$\boxed{\frac{d^2 X^i}{dT^2} = -R_{0i0j} X^j + \mathcal{O}\left(\left(\frac{|\vec{X}|}{\lambda}\right)^2\right)}$$

There are two transverse and torsional modes





$R_{0x0x} = -R_{0y0y}$ $\qquad$ R_{0x0y}

Oscillations are purely transverse (in the plane xy orthogonal to the direction of propagation z)

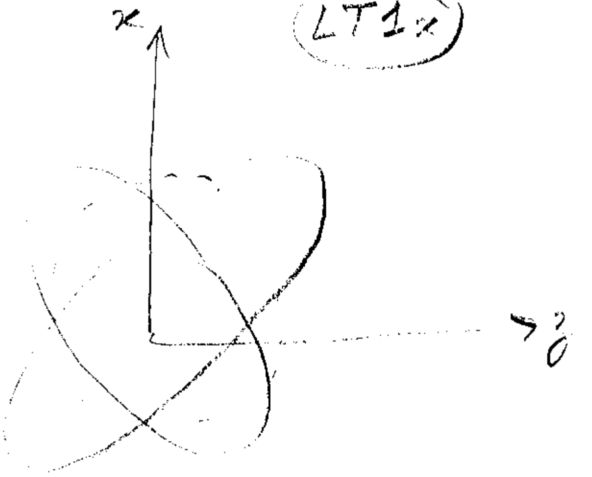
The deformation pattern remains invariant under a rotation along the direction of propagation z with angle π. Now from general consideration the invariance angle $\Delta\varphi$ is related to the spin of the wave by

$$\Delta\varphi = \frac{2\pi}{s}$$

Hence these two modes correspond to spin 2 (or tensorial) waves.

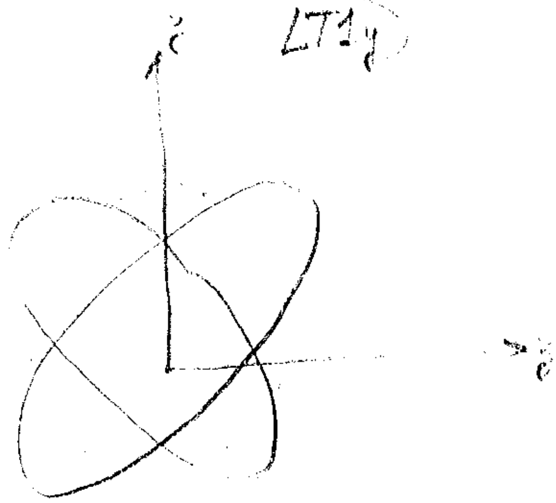
There are two vector modes

(LT1x)



R_{oxoz}

(LT1y)

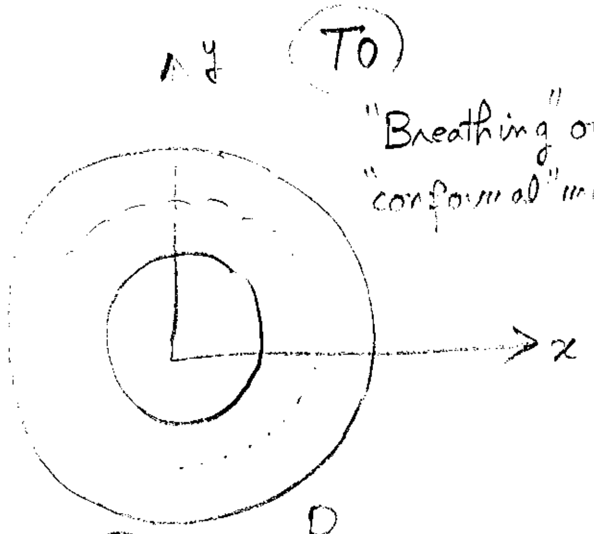


R_{oyoz}

These modes are both longitudinal and transverse and correspond to spin 1 (or vectorial) waves.

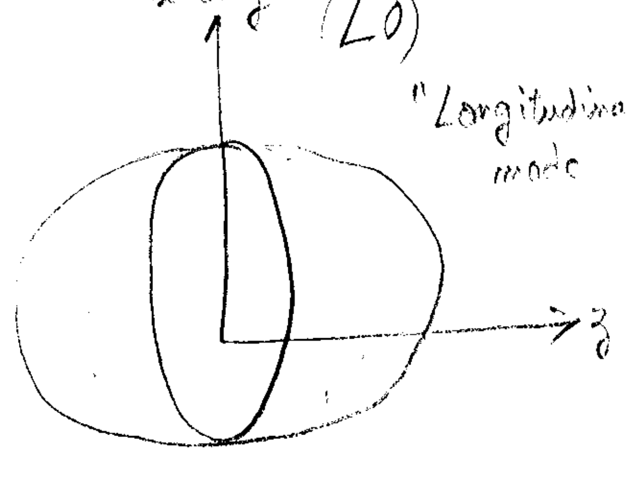
And there are two scalar modes, one transverse and one longitudinal,

(T0) "Breathing" or "conformal" mode



$R_{oxox} = R_{oyoy}$

(L0) "Longitudinal" mode



R_{ozoz}

Examples of metric theories of gravity

1. General Relativity (GR)

GR is the simplest metric theory of gravity, since the gravitational field is entirely described by the st. metric $g_{\mu\nu}$

It is based on the Einstein-Hilbert action ('915)

$$\boxed{S = \frac{c^3}{16\pi G}\int d^4x \sqrt{-g}\, \underset{\uparrow}{R} + S_m[\overset{\text{matter fields}}{\underset{\downarrow}{\psi_m}} ; g]}$$

scalar curvature

universal coupling of matter fields to the metric

We can also add a cosmological constant Λ (which plays a crucial role in cosmology today as being interpreted as the "dark energy")

$$S = \frac{c^3}{16\pi G}\int d^4x \sqrt{-g}\,(R - 2\Lambda) + S_m$$

The Einstein eqs (EE) obtained by variation de S (see later for the complete derivation) read

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

where $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ is the Einstein tensor and

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}$$

is the stress-energy tensor of the matter fields.

In vacuum, the Einstein eqs reduce to $\boxed{R_{\mu\nu} = 0}$ and this reduces the number of polarization modes.

Namely $R_{\mu\nu} = R^{\rho}_{\ \mu\rho\nu} \propto -R_{0\mu0\nu} + R_{x\mu x\nu} + R_{y\mu y\nu} + R_{z\mu z\nu} = 0$

and this gives 4 additional constraints namely

$$\underbrace{R_{0x0x} + R_{0y0y} = 0}_{\text{no 10 mode}} \qquad \underbrace{R_{0x0z} = R_{0y0z}}_{\text{no LT2 modes}} = \underbrace{R_{0z0z}}_{\text{no L0 mode}} = 0$$

and it remains only 2 polarization modes, namely the two spin-2 transverse modes T2+ and T2× corresponding to

$$R_{0x0x} + R_{0y0y} = 0 \quad \text{and} \quad R_{0x0y} = R_{0y0x}$$

GR corresponds to the propagation of a pure spin-2 wave, which is massless (since it propagates at speed c).

For a small perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

the field equations reduce at linear order to

$$\Box \bar{h}^{\mu\nu} - \partial^{\mu} H^{\nu} - \partial^{\nu} H^{\mu} + \eta^{\mu\nu} \partial_{\rho} H^{\rho} = -\frac{16\pi G}{c^4}\left(T^{\mu\nu} + t^{\mu\nu}\right)$$

where $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h$, $h = \eta^{\rho\sigma} h_{\rho\sigma}$, $H^{\mu} = \partial_{\nu} \bar{h}^{\mu\nu}$, $\Box = \eta^{\rho\sigma}\partial_{\rho}\partial_{\sigma}$ flat d'Alembertian operator

and $t^{\mu\nu} = O(h^2)$

These are admit the gauge invariance

$$\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu} + \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} - \eta^{\mu\nu}\partial_{\rho}\xi^{\rho}$$

which results from the diffeomorphism invariance of the full non-linear Einstein eqs (invariance by arbitrary change of coordinates $\{x\} \to \{x'\}$ with $x'^\mu = x^\mu + \xi^\mu(x)$). The gauge invariance of the linearized field eqs imply the conservation law*

$$\partial_\nu \left(\cdots T^{\mu\nu} + t^{\mu\nu} \right) = 0$$

which is fully equivalent to the covariant conservation of the matter stress-energy tensor

$$\nabla_\nu T^{\mu\nu} = 0$$

which follows from the EE by the Einstein identity $\nabla_\nu G^{\mu\nu} = 0$.

Using the gauge invariance we can make a choice of gauge and impose the so-called harmonic or de Donder condition $H^\mu = \partial_\nu \bar{h}^{\mu\nu} = 0$ so that the linearized field eqs read

$$\Box \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu} + O(h^2).$$

In vacuum $\Box \bar{h}^{\mu\nu} = 0$ and the theory admits wave solutions, e.g. $\bar{h}^{\mu\nu} = \bar{h}^{\mu\nu}(t - z/c)$.

2. Scalar-tensor theory (Jordan, Thiry, Fierz 1950'; Brans and Dicke 1960')

The gravitational field is described by the metric $g_{\mu\nu}$ (to which are coupled all matter fields) and by a scalar field ϕ.

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* It can be viewed as the conservation law associated with the gauge symmetry by Noether's theorem.

Most general scalar-tensor theory

$$\boxed{S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\left[F(\phi)R - Z(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2U(\phi)\right] + S_m[\psi_m; g_{\mu\nu}]}$$

Still all matter fields ψ_m are universally coupled to the metric $g_{\mu\nu}$ ("metric" coupling). There are 3 arbitrary functions of the scalar field $F(\phi)$, $Z(\phi)$, $U(\phi)$.* The scalar sector involves the scalar kinetic term $\propto g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$; and there is a potential term $\propto U(\phi)$. The factor $F(\phi)$ needs to be positive for the graviton to carry positive energy.

The metric $g_{\mu\nu}$ is called the physical or Jordan metric, and the previous action is written in the so-called Jordan frame. The Jordan metric $g_{\mu\nu}$ defines lengths and times as measured by laboratory rods and clocks (made of matter).

One can go to the so-called Einstein frame by a redefinition of variables. Namely

$$g^*_{\mu\nu} = F(\phi)\, g_{\mu\nu} \qquad \text{(conformal transformation)}$$

$$\left(\frac{d\varphi}{d\phi}\right)^2 = \frac{3}{4}\left(\frac{d\ln F}{d\phi}\right)^2 + \frac{Z}{2F} \qquad \text{(redefinition of scalar field } \phi \longrightarrow \varphi)$$

* One can always redefine the scalar field to be e.g. $F(\phi)$, so that after redefinition there are only 2 arbitrary functions.

$$2V(\varphi) = U(\Phi)F^{-2}(\Phi) \quad \text{(redefinition of potential)}$$

We also pose $A(\varphi) = (F(\Phi))^{-1/2}$ and $\alpha = \dfrac{d \ln A(\varphi)}{d\varphi}$.

Then the gravitational part reproduces exactly the Einstein-Hilbert action, while the kinetic term for the scalar field is standard,

$$S = \frac{1}{4\pi G}\int d^4x \sqrt{-g_*}\left(\frac{R^*}{4} - \frac{1}{2} g_*^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right) + S_m\left[\psi_m ; A^2(\varphi) g^*_{\mu\nu}\right]$$

In the Einstein frame, the theory is easy to analyze because the Einstein metric $g^*_{\mu\nu}$ is a pure spin-2 field and the scalar field φ is a pure spin-0 component.

The gravitational constant measured in a Cavendish experiment is not G but

$$\boxed{G_{measured} = G A^2(1 + \alpha^2)}$$

due to the exchange of a graviton between bodies

due to the exchange of a scalar particle

while the parametrized post-Newtonian (PPN) parameters tested in the Solar system and binary pulsars are

$$\gamma = 1 - 2\frac{\alpha^2}{1+\alpha^2}$$

$$\beta = 1 + \frac{\alpha^2}{2(1+\alpha^2)^2}\frac{d\alpha}{d\varphi}$$

Finally the theory has 3 polarization modes: in addition to the two spin-2 transverse modes T2+ and T2×, there is also the transverse scalar mode T0 (generated by the scalar field φ).

3. Massive gravity and bimetric theories

In GR the graviton is a particle with spin 2 and is massless (thus the range of the interaction is infinite).

Fierz and Pauli (1939) action: unique linear theory for a massive spin-2 field without "ghosts"

$$\boxed{S_{FP} = \frac{c^3}{16\pi G}\int d^4x \Big[\underbrace{\partial_\mu h_{\nu\rho}\partial^\mu \bar{h}^{\nu\rho} - H_\mu H^\mu}_{\text{Einstein-Hilbert action with } g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}} + \underbrace{\frac{m_g^2 c^2}{\hbar^2}\left(h_{\mu\nu}h^{\mu\nu} - h^2\right)}_{\text{mass term}}\Big] + \mathcal{O}(h^3)}$$

$\lambda_g = \frac{\hbar}{m_g c}$ Compton wavelength associated with graviton

Recently a massive gravity theory has been found (de Rham, Gabadadze 2010; de Rham, Gabadadze, Tolley 2010), which represents the unique extension of GR with a massive graviton, and is completely safe from the theoretical point of view, i.e. avoids ghosts at any order in perturbation theory.

In one version of it (Hassan and Rosen 2012) the theory is a bimetric theory, in which two metrics

$$g_{\mu\nu} \, ; \, f_{\mu\nu}$$

interact in a specific, uniquely fixed way. Thus, gravity is described by the physical metric $g_{\mu\nu}$ (to which all matter fields are coupled) and also by another metric $f_{\mu\nu}$. It is impossible to coupled the matter fields to both metrics at the same time.

$$\boxed{S_{\substack{\text{massive}\\ \text{gravity}}} = \int d^4x \Big[\underbrace{\frac{M_g^2}{2}\sqrt{-g}R_g}_{\substack{\text{kinetic term}\\ \text{of metric } g}} + \underbrace{m_g^2 V(g,f)}_{\substack{\text{interaction}\\ \text{term}}} + \underbrace{\frac{M_f^2}{2}\sqrt{-f}R_f}_{\substack{\text{kinetic term}\\ \text{of metric } f}} \Big]}$$

Define $X = \sqrt{g^{-1}f}$ i.e. $X^\mu_\rho X^\rho_\nu = g^{\mu\rho} f_{\nu\rho}$

and $Y = \sqrt{f^{-1}g}$ $\qquad Y^\mu_\rho Y^\rho_\nu = f^{\mu\rho} g_{\nu\rho}$

$$V = \sqrt{-g}\sum_{n=0}^{4} \alpha_n e_n(X) = \sqrt{-f}\sum_{n=0}^{4} \beta_n e_n(Y)$$

5 numerical constants

where $e_n(X)$ are the symmetric polynomials associated with the square-root matrix X^μ_ν:

$$e_0(X) = -\frac{1}{24}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma}$$

$$e_1(X) = -\frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\lambda}X^\lambda_\sigma$$

$$e_2(X) = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\tau\lambda}X^\tau_\rho X^\lambda_\sigma$$

$$\vdots$$

This theory has <u>5 polarization modes</u>:

T2+, T2X : spin-2 transverse modes like in GR

LT1x, LT1y : spin-1 longitudinal modes

T0 : spin-0 transverse mode

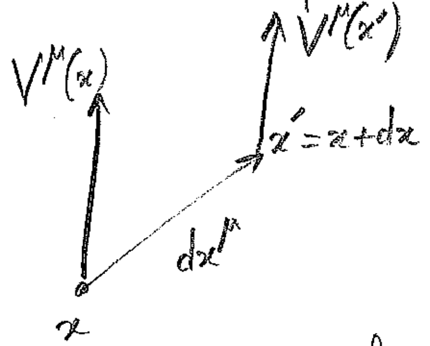
But <u>the longitudinal spin-0 mode L0 is forbidden in this theory, this is a ghost</u>.

Einstein-Carton theory: an example of non-metric theory of gravity

It is based on a Riemann-Cartan space-time manifold.

Basically we keep the notion of parallel transport

For a vector field $V^\mu(x)$ parallely transported from x to x'



$\exists$ affine connection $\Gamma^{\mu}_{\cdot\,\nu\rho}$ such that

$$\boxed{dV^\mu = -\Gamma^{\mu}_{\cdot\,\nu\rho}\, dx^\nu V^\rho}$$

where $dV^\mu = V^\mu(x+dx) - V^\mu(x) = dx^\nu \partial_\nu V^\mu$

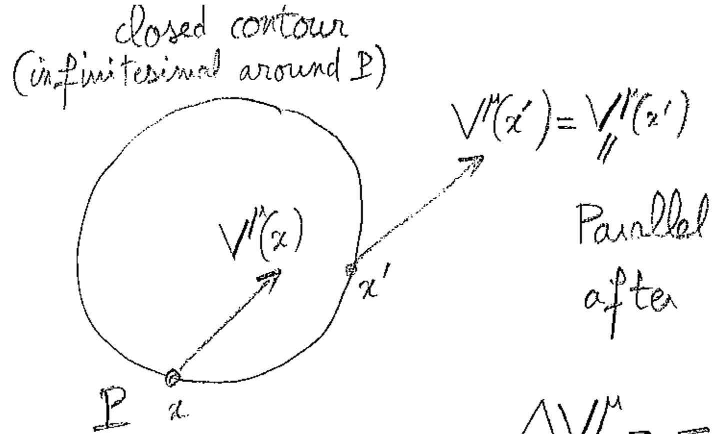
In a coord. transformation $\{x\} \to \{x'\}$ this connection transforms like the Christoffel symbol

$$\Gamma'^{\mu}_{\cdot\,\nu\rho}(x') = \frac{\partial x'^\mu}{\partial x^\varepsilon} \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x^\sigma}{\partial x'^\rho} \Gamma^{\varepsilon}_{\cdot\,\lambda\sigma} + \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\nu \partial x'^\rho}$$

But $\Gamma^{\mu}_{\cdot\,\nu\rho}$ is not necessarily symmetric in $\nu\rho$. However its anti-symmetric part is a tensor, the Cartan tensor or torsion

$$\boxed{C^{\mu}_{\cdot\,\nu\rho} = \Gamma^{\mu}_{\cdot\,[\nu\rho]}}$$

The curvature is defined by the parallel transport of a vector along an infinitesimal closed contour (exactly like in GR)



Parallel transport equation gives after a round trip

$$\Delta V^\mu = - \oint \Gamma^\mu_{\cdot\nu\rho}\, dx^\nu\, V^\rho$$

Near P
(Taylor expansion to first order)

$$\Gamma^\mu_{\cdot\nu\rho}(x) = \Gamma^\mu_{\cdot\nu\rho}(x_P) + (x^\sigma - x_P^\sigma)\,\partial_\sigma \Gamma^\mu_{\cdot\nu\rho}(x_P) + \mathcal{O}\left(|x-x_P|^2\right)$$

and the sol. of the parallel transport equation to first order is

$$V^\rho(x) = V^\rho(x_P) - (x^\sigma - x_P^\sigma)\,\Gamma^\rho_{\cdot\sigma\lambda}(x_P)\, V^\lambda(x_P) + \mathcal{O}\left(|x-x_P|^2\right)$$

$$\Delta V^\mu = - \oint \Big[\Gamma^\mu_{\cdot\nu\rho}(x_P)\, V^\rho(x_P) - (x^\sigma - x_P^\sigma)\,\Gamma^\rho_{\cdot\sigma\lambda}\,\Gamma^\mu_{\cdot\nu\rho}\, V^\lambda + (x^\sigma - x_P^\sigma)\,\partial_\sigma \Gamma^\mu_{\cdot\nu\rho}\, V^\rho + \mathcal{O}(2) \Big] dx^\nu$$

But $\oint dx^\nu = 0$ around a closed contour

$$\Delta V^\mu = - \left[\partial_\sigma \Gamma^\mu_{\cdot\nu\rho}\, V^\rho - \Gamma^\rho_{\cdot\sigma\lambda}\,\Gamma^\mu_{\cdot\nu\rho}\, V^\lambda \right](x_P)\, \underbrace{\oint x^\sigma dx^\nu}$$

From Stoke's theorem $\oint x^\sigma dx^\nu = \int_{\text{surface limited by the contour}} dx^\sigma \wedge dx^\nu$

NB: Stoke's theorem is best expressed in terms of differential forms

If $f = f_\mu dx^\mu$ then $df = \partial_\mu f_\nu\, dx^\mu \wedge dx^\nu$ and we have

$$\boxed{\oint_{\text{contour}} f = \int_{\text{surface}} df}$$

We define (like in GR)

$$\boxed{\Delta V^\mu = -\frac{1}{2} R^\mu_{\circ\nu\rho\sigma} V^\nu \int_{\text{surface}} dx^\rho \wedge dx^\sigma}$$

(for an infinitesimal contour)

$$\boxed{R^\mu_{\circ\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\circ\sigma\nu} - \partial_\sigma \Gamma^\mu_{\circ\rho\nu} + \Gamma^\mu_{\circ\rho\lambda} \Gamma^\lambda_{\circ\sigma\nu} - \Gamma^\mu_{\circ\sigma\lambda} \Gamma^\lambda_{\circ\rho\nu}}$$

(antisymmetric w.r.t. 2^d pair of indices $\rho\sigma$)

On the Riemann–Cartan space-time we have a covariant derivative defined by

$dx^\nu \nabla_\nu V^\mu = 0$ for parallely transported vectors

Hence $$\boxed{\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\circ\nu\rho} V^\rho}$$

index of derivation comes first

The s.t. is metric, and the metric $g_{\mu\nu}$ (symetric in $\mu\nu$) is defined to be compatible with the covariant derivative

$$\boxed{\nabla_\lambda g_{\mu\nu} = 0}$$

$$\partial_\lambda g_{\mu\nu} - \cancel{\Gamma^\rho_{\circ\lambda\mu} g_{\nu\rho}} - \Gamma^\rho_{\circ\lambda\nu} g_{\mu\rho} = 0$$

$$\partial_\nu g_{\mu\lambda} - \cancel{\Gamma^\rho_{\circ\nu\mu} g_{\lambda\rho}} - \Gamma^\rho_{\circ\nu\lambda} g_{\mu\rho} = 0$$

$$-\partial_\mu g_{\nu\lambda} + \cancel{\Gamma^\rho_{\circ\mu\nu} g_{\lambda\rho}} + \cancel{\Gamma^\rho_{\circ\mu\lambda} g_{\nu\rho}} = 0$$

$$0 = \partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\nu\lambda} - 2\, C^{\rho}_{\ \cdot\lambda\mu}\, g_{\nu\rho}$$
$$- 2\, C^{\rho}_{\ \cdot\nu\mu}\, g_{\lambda\rho} - 2\, \Gamma^{\rho}_{\ \cdot(\lambda\nu)}\, g_{\mu\rho}$$

Define the usual Christoffel symbol as

$$\boxed{\left\{ {\mu \atop \nu\rho} \right\} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} \right)}$$

$$0 = \left\{ {\mu \atop \nu\lambda} \right\} - C_{\nu\lambda\cdot}^{\ \ \ \mu} - C_{\lambda\nu\cdot}^{\ \ \ \mu} - \Gamma^{\mu}_{\ \cdot(\nu\lambda)}$$

$$\Gamma^{\mu}_{\ \cdot\nu\lambda} = \Gamma^{\mu}_{\ \cdot(\nu\lambda)} + \Gamma^{\mu}_{\ \cdot[\nu\lambda]}$$
$$= \left\{ {\mu \atop \nu\lambda} \right\} + C_{\nu\cdot\lambda}^{\ \mu} + C_{\lambda\cdot\nu}^{\ \mu} + \Gamma^{\mu}_{\ \cdot[\nu\lambda]}$$

$$\boxed{\Gamma^{\mu}_{\ \cdot\nu\lambda} = \left\{ {\mu \atop \nu\lambda} \right\} + K^{\mu}_{\ \cdot\nu\lambda}}$$

where the tensor $K^{\mu}_{\ \cdot\nu\lambda}$ is called the <u>contorsion</u>

$$\boxed{K^{\mu}_{\ \cdot\nu\lambda} = C_{\nu\cdot\lambda}^{\ \mu} + C_{\lambda\cdot\nu}^{\ \mu} + C^{\mu}_{\ \cdot\nu\lambda}}$$

It is anti-symmetric in the 1st and 3d indices

$$\boxed{K_{\mu\nu\lambda} = - K_{\lambda\nu\mu}}$$

On a Riemann – Cartan space-time the <u>Einstein equivalence principle (EEP) is not valid</u>*. One cannot define a locally inertial frame $\{X^\alpha\}$ such that $g_{\alpha\beta}(X)$ equals $\eta_{\alpha\beta}$ at a point P and differs around P by small corrections of second order in the distance.

If $g_{\alpha\beta} = \eta_{\alpha\beta} + \mathcal{O}(2)$ then $\partial_\gamma g_{\alpha\beta} = \mathcal{O}(1)$ hence $\left\{ {\gamma \atop \alpha\beta} \right\} = \mathcal{O}(1)$

But by def. of the covariant derivative $\nabla_\gamma g_{\alpha\beta} = 0 = 2\Gamma^{\delta}_{\cdot\gamma(\alpha} g_{\beta)\delta}$ hence we should have $\Gamma = K$ which cannot be zero. There is a non trivial parallel transport.

One can thus erect loc. inertial coord. (and thus recover local Lorentz invariance LLI) only if the contorsion and torsion are zero.

The Ricci tensor $R_{\mu\nu} = R^{\lambda}_{\cdot\mu\lambda\nu}$ is no longer symmetric

$$\boxed{R_{[\mu\nu]} = (\nabla_\rho + T_\rho)\, T^{\rho}_{\cdot\nu\mu}}$$

where $T^{\rho}_{\cdot\nu\mu} = C^{\rho}_{\cdot\nu\mu} + 2\,\delta^{\rho}_{[\nu}\, C^{\lambda}_{\cdot\mu]\lambda}$ is the <u>modified torsion</u>

and $T_\rho = T^{\sigma}_{\cdot\sigma\rho}$. The Riemann tensor $R_{\mu\nu\rho\sigma}$ is still anti-symmetric wrt $\mu\nu$ and $\rho\sigma$ but the cyclic symmetry is modified. Bianchi identities are modified too, as well as Ricci identity

$$\boxed{(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu) V^\rho = R^{\rho}_{\cdot\sigma\mu\nu} V^\sigma - 2\, C^{\sigma}_{\cdot\mu\nu} \nabla_\sigma V^\rho}$$

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* Hence the theory is not metric.

Einstein – Cartan theory is defined by the same action as GR

$$\boxed{S = \frac{1}{16\pi}\int d^4x \sqrt{-g}\, R \; + S_m[\Psi, g, C]}$$

where R is the scalar curvature (depending on metric and torsion) and the matter action depend also on the torsion.

To get a non-trivial theory we need that the matter tensor depends on $C^{\mu}_{\cdot\nu\rho}$. This dependence will be through a covariant derivative in the matter action, so we need particles with spins. Indeed, matter with spins implies derivatives

$$\rho(\vec{x}) \sim m\,\delta_{(3)}(\vec{x}) + \frac{1}{c} S^i \partial_i \delta_{(3)}(\vec{x}) + \cdots$$

and the minimal coupling to gravity $\partial_i \longrightarrow \nabla_\mu$ will imply covariant derivatives and hence a dependence on torsion. See the description of particles with spins.

The field equations are (exercice)

$$\boxed{\begin{aligned} G_{\mu\nu} + (\nabla_\rho + T_\rho)\, U_{\mu\nu\cdot}^{\;\;\;\;\rho} &= 8\pi\, T_{\mu\nu} \\ U^{\mu\;\;\nu}_{\cdot\rho\cdot} &= 8\pi\, \Sigma_\rho^{\cdot\mu\nu} \end{aligned}}$$

where $U^{\mu\;\;\nu}_{\cdot\rho\cdot} = T_{\rho\cdot\cdot}^{\;\;\mu\nu} + T^{\mu\;\;\nu}_{\cdot\rho\cdot} + T^{\nu\mu}_{\cdot\cdot\rho}$ and

$$\boxed{T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \qquad \Sigma_\rho^{\cdot\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_m}{\delta C^\rho_{\cdot\mu\nu}}}$$

If S_m does not depend on $C^\rho_{\cdot\mu\nu}$ Einstein – Cartan simply reduces to Einstein.