Ecole de Gif 2017

Relativité générale et ondes gravitationnelles

APPROXIMATION METHODS IN GR AND GRAVITATIONAL WAVES

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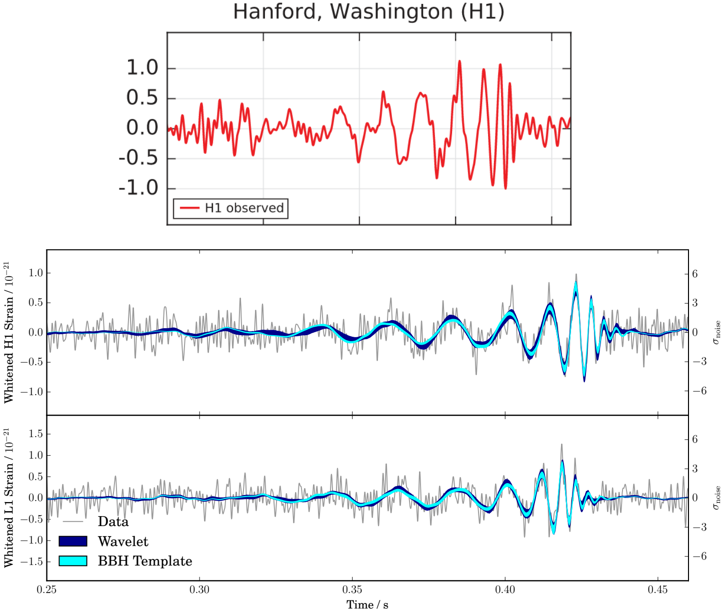
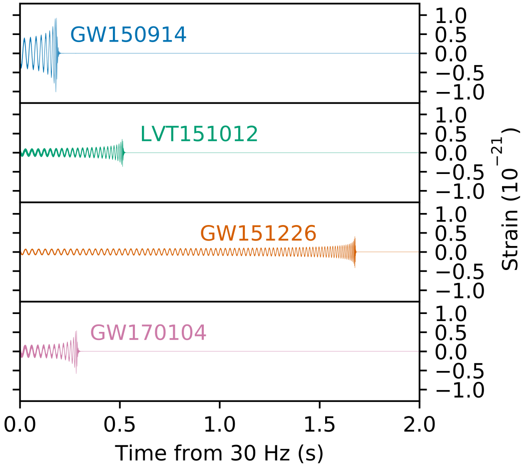
Binary black-hole event GW150914 [LIGO/VIRGO collaboration 2016]

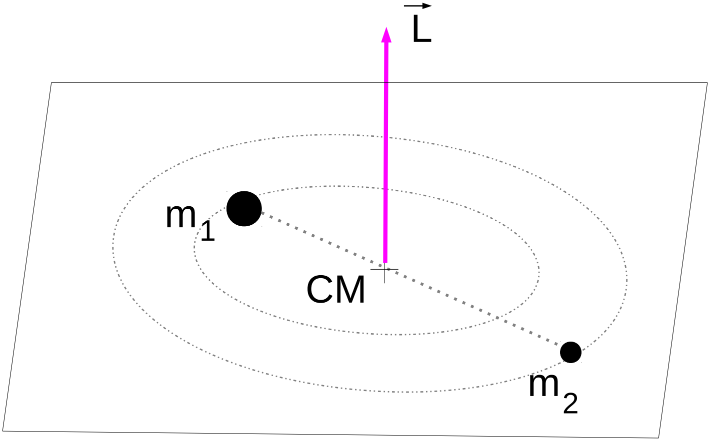


Gravitational wave BBH events [LIGO/VIRGO collaboration 2016, 2017]

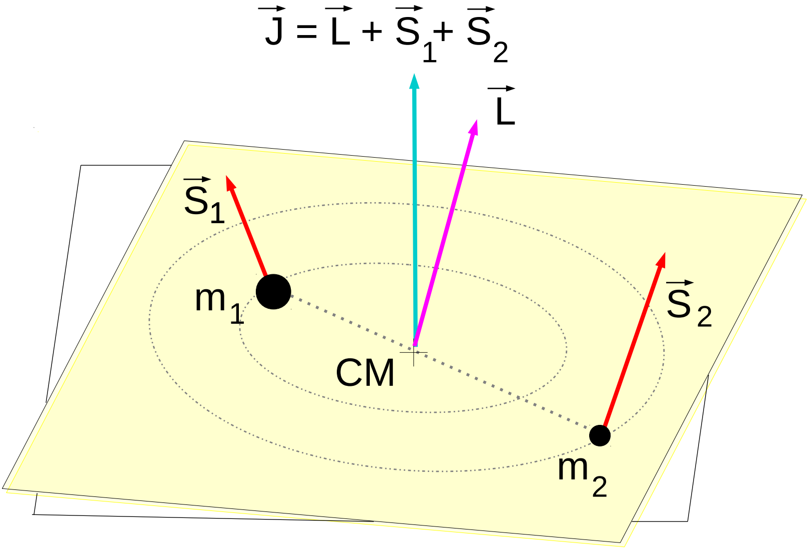


For BH binaries the detectors are mostly sensitive to the merger phase and a few cycles are observed before coalescence

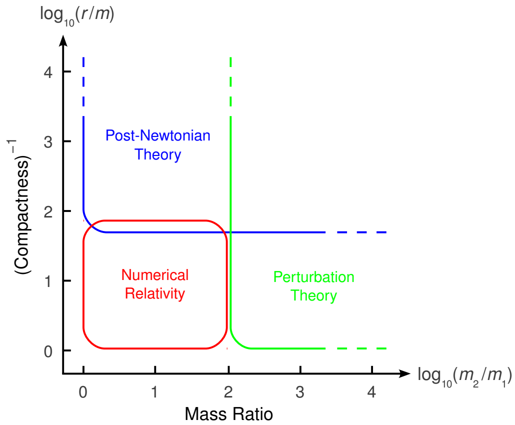
Modelling the compact binary dynamics



Modelling the compact binary dynamics



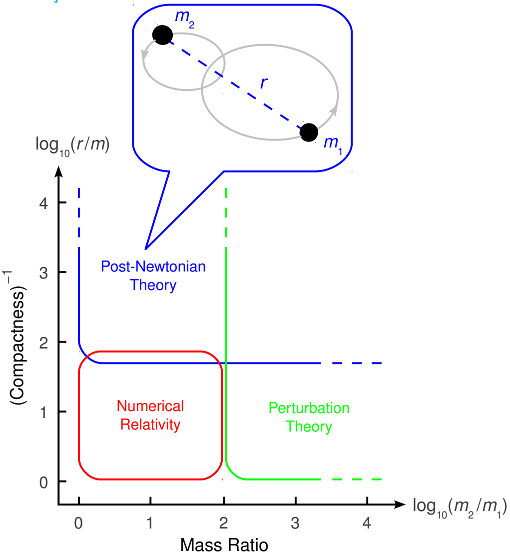
Methods to compute GW templates



[courtesy Alexandre Le Tiec]

Methods to compute GW templates

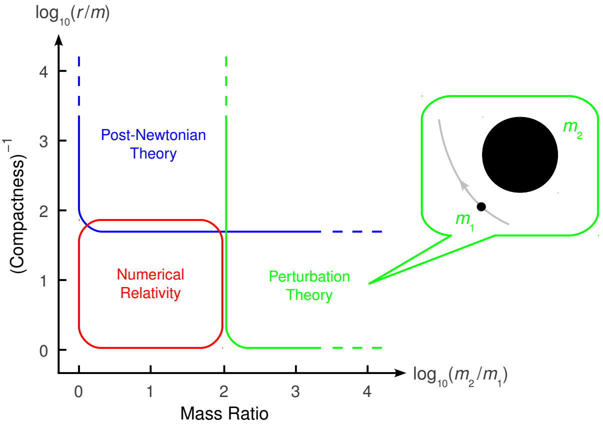
[see Blanchet 2014 for a review]



[courtesy Alexandre Le Tiec]

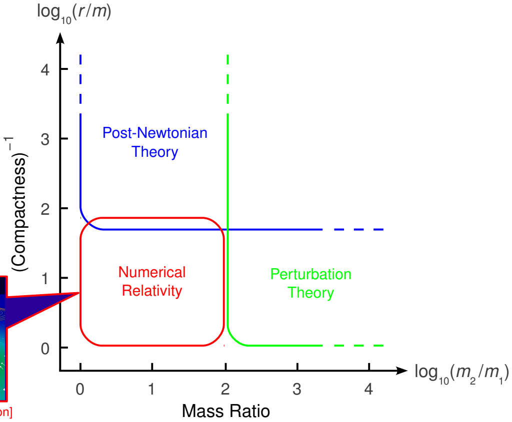
Methods to compute GW templates

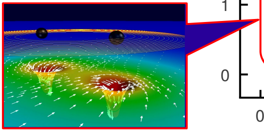
[Detweiler 2008; Barack 2009]



[courtesy Alexandre Le Tiec]

Methods to compute GW templates



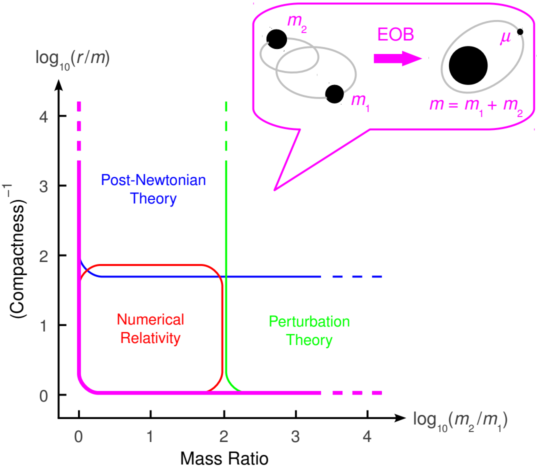


[Caltech/Cornell/CITA collaboration]

[courtesy Alexandre Le Tiec]

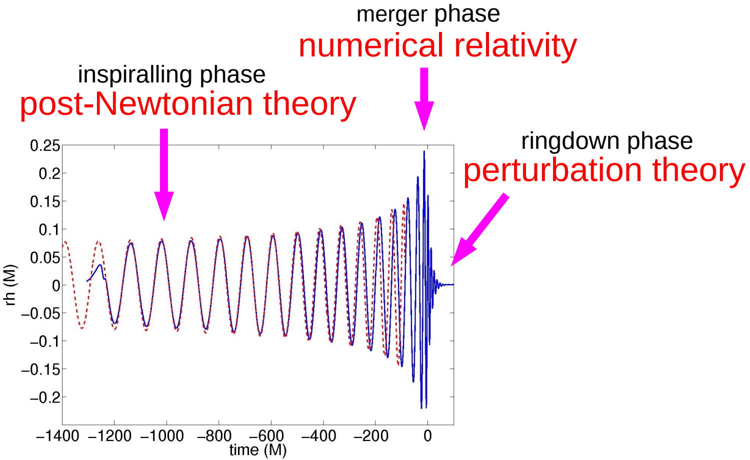
Methods to compute GW templates

[Buonanno & Damour 1998]



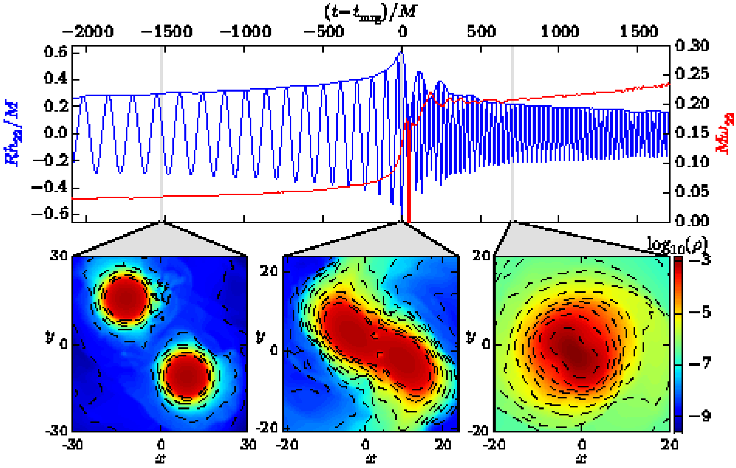
[courtesy Alexandre Le Tiec]

The gravitational chirp of compact binaries



Post-merger waveform of neutron star binaries

[Shibata *et al.*, Rezzolla *et al.* 1990-2010s]



Methods to compute PN equations of motion

1. ADM Hamiltonian canonical formalism [Ohta *et al.* 1973; Schäfer 1985]
2. EOM in harmonic coordinates [Damour & Deruelle 1985; Blanchet & Faye 1998, 2000]
3. Extended fluid balls [Grishchuk & Kopeikin 1986]
4. Surface-integral approach [Itoh, Futamase & Asada 2000]
5. Effective-field theory (EFT) [Goldberger & Rothstein 2006; Foffa & Sturani 2011]

- EOM derived in a general frame for arbitrary orbits
- Dimensional regularization is applied for UV divergences[1]
- Radiation-reaction dissipative effects added separately by matching
- Spin effects can be computed within a pole-dipole approximation
- Tidal effects incorporated at leading 5PN and sub-leading 6PN orders

[1] Except in the surface-integral approach

Methods to compute PN radiation field

1. Multipolar-post-Minkowskian (MPM) & PN [Blanchet-Damour-Iyer 1986, . . . , 1998]
2. Direct iteration of the relaxed field equations (DIRE) [Will-Wiseman-Pati 1996, . . .]
3. Effective-field theory (EFT) [Hari Dass & Soni 1982; Goldberger & Ross 2010]

- Involves a machinery of tails and related non-linear effects
- Uses dimensional regularization to treat point-particle singularities
- Phase evolution relies on balance equations valid in adiabatic approximation
- Spin effects are incorporated within a pole-dipole approximation
- Provides polarization waveforms for DA & spin-weighted spherical harmonics decomposition for NR

Gauge-fixed Einstein field equations

- Add the standard harmonic coordinates gauge-fixing term

$$S_{\text{gauge-fixed}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x \left(\sqrt{-g}\,R \underbrace{- \frac{1}{2}\mathfrak{g}_{\alpha\beta}\partial_\mu\mathfrak{g}^{\alpha\mu}\partial_\nu\mathfrak{g}^{\beta\nu}}_{\text{gauge-fixing term}}\right) + S_{\text{mat}}$$

where $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$ is the so-called ghotic metric

$$\mathfrak{g}^{\mu\nu}\partial^2_{\mu\nu}\mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Sigma^{\alpha\beta}[\mathfrak{g},\partial\mathfrak{g}]}^{\text{non-linear source term}}$$

$$\underbrace{\partial_\mu\mathfrak{g}^{\alpha\mu} = 0}_{\text{harmonic-gauge condition}}$$

- Such system of equations constitutes a well-posed problem ("problème bien posé") in the sense of Hadamard [Choquet-Bruhat 1952]

Perturbation around Minkowski space-time

- Assume the space-time slightly differs from Minkowski space-time $\eta_{\alpha\beta}$

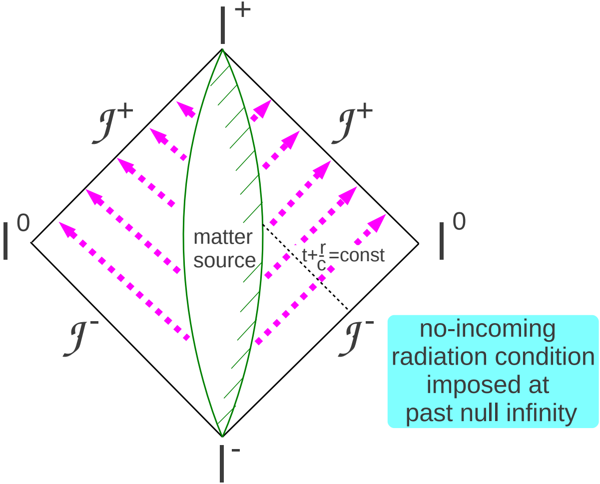
$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta} \qquad \text{with} \quad |h| \ll 1$$

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Lambda^{\alpha\beta}[h, \partial h, \partial^2 h]}^{\text{non-linear source term}} \equiv \frac{16\pi G}{c^4} \underbrace{\tau^{\alpha\beta}}_{\text{stress-energy pseudo-tensor}}$$

$$\underbrace{\partial_\mu h^{\alpha\mu} = 0}_{\text{harmonic-gauge condition}}$$

- Such system can be resolved assuming Minkowskian boundary conditions of no incoming radiation imposed at at $\mathcal{I}^-$

No-incoming radiation condition



$$\lim_{\substack{r\to+\infty\\ t+\frac{r}{c}=\text{const}}}\left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(rh^{\alpha\beta}\right)=0$$

Post-Minkowskian expansion [e.g. Bertotti & Plebanski 1960]

- Appropriate for weakly self-gravitating isolated matter sources

$$\gamma_{\text{PM}} \equiv \frac{GM}{c^2 a} \ll 1 \qquad \begin{cases} M & \text{mass of source} \\ a & \text{size of source} \end{cases}$$

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + \underbrace{\sum_{n=1}^{+\infty} G^n \, h_{(n)}^{\alpha\beta}}_{G \text{ labels the PM expansion}}$$

$$\Box h_{(n)}^{\alpha\beta} = \frac{16\pi G}{c^4} |g| T_{(n)}^{\alpha\beta} + \overbrace{\Lambda_{(n)}^{\alpha\beta}[h_{(1)}, \cdots, h_{(n-1)}]}^{\text{know from previous iterations}}$$

$$\partial_\mu h_{(n)}^{\alpha\mu} = 0$$

- Very difficult approximation to implement in practice for general sources at high PM orders [Thorne & Kovàcs 1975]

Post-Newtonian expansion

[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938; Fock 1959; Chandrasekhar 1965; Will 1972; *etc.*]

Valid for isolated matter sources that are at once slowly moving, weakly stressed and weakly gravitating (so-called post-Newtonian source) in the sense that

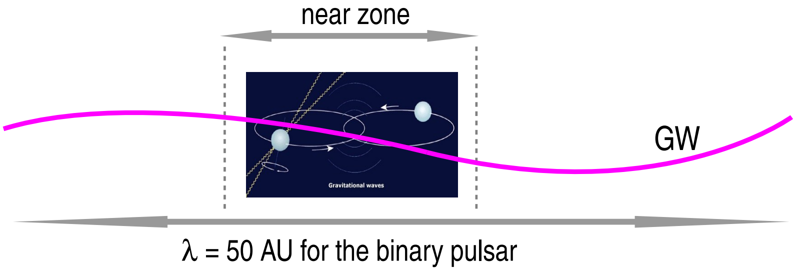
$$\boxed{\varepsilon_{\text{PN}} \equiv \max\left\{\left|\frac{T^{0i}}{T^{00}}\right|, \left|\frac{T^{ij}}{T^{00}}\right|^{1/2}, \left|\frac{U}{c^2}\right|^{1/2}\right\} \ll 1}$$

- ε_{PN} plays the role of a slow motion estimate $\varepsilon_{\text{PN}} \sim v/c \ll 1$
- For self-gravitating sources the internal motion is due to gravitational forces (*e.g.* a Newtonian binary system) hence $v^2 \sim GM/a$
- Gravitational wavelength $\lambda \sim cP$ where $P \sim a/v$ is the period of motion

$$\boxed{\frac{a}{\lambda} \sim \frac{v}{c} \sim \varepsilon_{\text{PN}}}$$

Post-Newtonian expansion

[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1932; Fock 1959; Chandrasekhar 1965; Will 1972; *etc.*]



- Near zone defined by $r \ll \lambda$ covers entirely the post-Newtonian source
- General PN expansion inside the source's near zone

$$h^{\alpha\beta}_{\mathrm{PN}}(\mathbf{x}, t, c) = \sum_{p\geqslant 2} \frac{1}{c^p} h^{\alpha\beta}_p(\mathbf{x}, t, \ln c)$$

Multipolar expansion [e.g. Pirani 1964; Thorne 1980]

- Valid in the exterior of any possibly strong field isolated source

$$\frac{a}{r} < 1 \qquad \begin{cases} a \text{ size of source} \\ r \text{ distance to source} \\ \lambda \sim cP \text{ wavelength of radiation} \end{cases}$$

$$\underbrace{M_L \sim M a^\ell}_{\text{mass-type multipole moment}} \qquad \underbrace{S_L \sim M a^\ell v}_{\text{current-type multipole moment}} \qquad (L = i_1 \cdots i_\ell)$$

- Split space-time into near zone $r \ll \lambda$ and wave zone $r \gg \lambda$

$$\underbrace{h_{\text{NZ}} \sim \frac{G}{c^2} \sum_\ell \left[\frac{M_L}{r^{\ell+1}} + \frac{S_L}{c r^{\ell+1}} \right]}_{r \ll \lambda} \qquad \underbrace{h_{\text{WZ}} \sim \frac{G}{c^2 r} \sum_\ell \left[\frac{M_L^{(\ell)}}{c^\ell} + \frac{S_L^{(\ell)}}{c^{\ell+1}} \right]}_{r \gg \lambda}$$

- The multipole expansion viewed in the WZ is a post-Newtonian expansion

Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]

$$4\pi R^2 \bar{\mathfrak{J}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^{\,2} - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right].$$

1. Einstein quadrupole formula

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\text{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}$$

2. Amplitude quadrupole formula

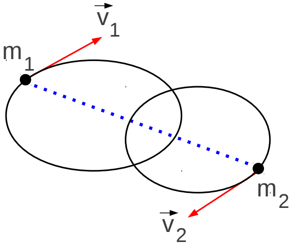
$$h_{ij}^{\text{TT}} = \frac{2G}{c^4 D}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t - \frac{D}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\text{TT}} + \mathcal{O}\left(\frac{1}{D^2}\right)$$

3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$F_i^{\text{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

Application to compact binaries [Peters & Mathews 1963; Peters 1964]



$$\begin{cases} a \text{ semi-major axis of relative orbit} \\ e \text{ eccentricity of relative orbit} \\ \omega = \frac{2\pi}{P} \text{ orbital frequency} \end{cases}$$

$$M = m_1 + m_2$$
$$\mu = \frac{m_1 m_2}{M}$$

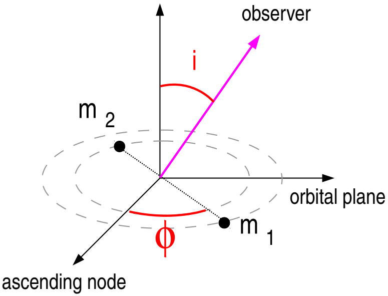
$$\boxed{\nu = \frac{\mu}{M} \qquad 0 < \nu \leqslant \frac{1}{4}}$$

$$\langle \mathcal{F}^{\text{GW}} \rangle = \frac{32}{5} \frac{c^5}{G} \nu^2 \left(\frac{GM}{ac^2} \right)^5 \underbrace{\frac{1 + \frac{73}{24} e^2 + \frac{37}{96} e^4}{(1 - e^2)^{7/2}}}_{\text{"enhancement" factor } f(e)}$$

Energy balance argument $\frac{\mathrm{d}E}{\mathrm{d}t} = -\langle \mathcal{F}^{\text{GW}} \rangle$ together with Kepler's law $GM = a^3 \omega^2$

$$\boxed{\dot{P} = -\frac{192\pi}{5c^5} \left(\frac{2\pi GM}{P} \right)^{5/3} \nu \, f(e)}$$

Waveform of inspiralling compact binaries



$$h_+ = \frac{2G\mu}{c^2 D_\mathsf{L}} \left(\frac{GM\omega}{c^3}\right)^{2/3} \left(1+\cos^2 i\right) \cos\left(2\phi\right)$$

$$h_\times = \frac{2G\mu}{c^2 D_\mathsf{L}} \left(\frac{GM\omega}{c^3}\right)^{2/3} \left(2\cos i\right) \sin\left(2\phi\right)$$

The distance of the source $r = D_\mathsf{L}$ is measurable from the GW signal [Schutz 1986]

Orbital phase of inspiralling compact binaries

for quasi-circular orbits
$$\begin{cases} E = -\dfrac{Mc^2}{2}\nu\, x \\ \mathcal{F}^{\mathsf{GW}} = \dfrac{32}{5}\dfrac{c^5}{G}\nu^2 x^5 \end{cases}$$

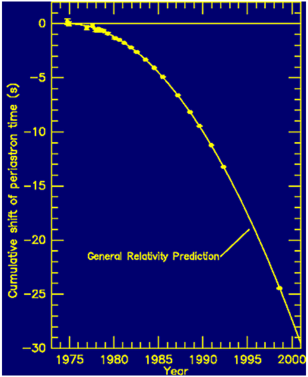
where $x = \left(\dfrac{GM\omega}{c^3}\right)^{2/3}$ = PN parameter = $\mathcal{O}(\varepsilon_{\mathsf{PN}}^2)$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}^{\mathsf{GW}} \iff \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{64}{5}\frac{c^3\nu}{GM}\,x^5 \iff \frac{\dot{\omega}}{\omega^2} = \frac{96\nu}{5}\nu\left(\frac{GM\omega}{c^3}\right)^{5/3}$$

$$\boxed{\begin{aligned} a(t) &= \left(\frac{256}{5}\frac{G^3M^3\nu}{c^5}(t_c - t)\right)^{1/4} \\ \phi(t) &= \phi_c - \frac{1}{32\nu}\left(\frac{256}{5}\frac{c^3\nu}{GM}(t_c - t)\right)^{5/8} \end{aligned}}$$

The quadrupole formula works for the binary pulsar

[Taylor & Weisberg 1982]



$$\dot{P} = -\frac{192\pi}{5c^5}\nu\left(\frac{2\pi G\,M}{P}\right)^{5/3}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \approx -2.4\times 10^{-12}$$

[Peters & Mathews 1963; Esposito & Harrison 1975; Wagoner 1975; Damour & Deruelle 1983]

The quadrupole formula works also for GW150914 !

1. The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi}\left[\frac{256}{5}\frac{G\mathcal{M}^{5/3}}{c^5}(t_\mathsf{f} - t)\right]^{-3/8}$$

2. Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96}\frac{c^5}{G\pi^{8/3}} f^{-11/3}\dot{f}\right]^{3/5}}$$

which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

3. The GW amplitude is predicted to be

$$h_\mathsf{eff} \sim 4.1\times 10^{-22}\left(\frac{\mathcal{M}}{M_\odot}\right)^{5/6}\left(\frac{100\,\mathsf{Mpc}}{D}\right)\left(\frac{100\,\mathsf{Hz}}{f_\mathsf{merger}}\right)^{-1/6} \sim 1.6\times 10^{-21}$$

4. The distance $D = 400\,\mathsf{Mpc}$ is measured from the signal itself

Total energy radiated by GW150914

1. The ADM energy of space-time is constant and reads (at any t)

$$E_{\text{ADM}} = (m_1 + m_2)c^2 - \frac{G m_1 m_2}{2r} + \frac{G}{5c^5}\int_{-\infty}^{t} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t')$$

2. Initially $E_{\text{ADM}} = (m_1 + m_2)c^2$ while finally (at time t_{f})

$$E_{\text{ADM}} = M_{\mathsf{f}} c^2 + \frac{G}{5c^5}\int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t')$$

3. The total energy radiated in GW is

$$\boxed{\Delta E^{\text{GW}} = (m_1 + m_2 - M_{\mathsf{f}})c^2 = \frac{G}{5c^5}\int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t') = \frac{G m_1 m_2}{2 r_{\mathsf{f}}}}$$

4. The total power released is

$$\boxed{P^{\text{GW}} \sim \frac{3 M_\odot c^2}{0.2\,\mathsf{s}} \sim 10^{49}\,\mathsf{W} \sim 10^{-3}\,\frac{c^5}{G}}$$

Linearized multipolar vacuum solution [Thorne 1980]

General solution of linearized vacuum field equations in harmonic coordinates

$$\boxed{\Box h^{\alpha\beta}_{(1)} = \partial_\mu h^{\alpha\mu}_{(1)} = 0}$$

$$h^{00}_{(1)} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}M_L(u)\right)$$

$$h^{0i}_{(1)} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}M^{(1)}_{iL-1}(u)\right) + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}S_{bL-1}(u)\right)\right\}$$

$$h^{ij}_{(1)} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}M^{(2)}_{ijL-2}(u)\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\frac{1}{r}\epsilon_{ab(i}S^{(1)}_{j)bL-2}(u)\right)\right\}$$

- multipole moments $M_L(u)$ and $S_L(u)$ arbitrary functions of $u = t - r/c$
- mass $M = \text{const}$, center-of-mass position $X_i \equiv M_i/M = \text{const}$, linear momentum $P_i \equiv M^{(1)}_i = 0$, angular momentum $S_i = \text{const}$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

- The linearized solution is the starting point of an explicit MPM algorithm

$$h^{\alpha\beta}_{\mathrm{MPM}} = \sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)}$$

- Hierarchy of perturbation equations is solved by induction over n

$$\Box h^{\alpha\beta}_{(n)} = \Lambda^{\alpha\beta}_{(n)}[h_{(1)}, h_{(2)}, \ldots, h_{(n-1)}]$$
$$\partial_\mu h^{\alpha\mu}_{(n)} = 0$$

- A regularization is required in order to cope with the divergency of the multipolar expansion when $r \to 0$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

1. Multiply source term by r^B where $B \in \mathbb{C}$ and integrate

$$u^{\alpha\beta}_{(n)}(B) = \Box^{-1}_{\text{ret}} \left[r^B \Lambda^{\alpha\beta}_{(n)} \right]$$

2. Consider Laurent expansion when $B \to 0$

$$u^{\alpha\beta}_{(n)}(B) = \sum_{j=j_{\text{min}}}^{+\infty} u^{\alpha\beta}_{j(n)} B^j \quad \text{then} \quad \begin{cases} j < 0 & \Longrightarrow \quad \Box u^{\alpha\beta}_{j(n)} = 0 \\ j \geqslant 0 & \Longrightarrow \quad \Box u^{\alpha\beta}_{j(n)} = \frac{(\ln r)^j}{j!} \Lambda^{\alpha\beta}_{(n)} \end{cases}$$

3. Define the finite part (FP) when $B \to 0$ to be the zeroth coefficient $u^{\alpha\beta}_{0(n)}$

$$u^{\alpha\beta}_{(n)} = \text{FP} \Box^{-1}_{\text{ret}} \left[r^B \Lambda^{\alpha\beta}_{(n)} \right] \quad \text{then} \quad \Box u^{\alpha\beta}_{(n)} = \Lambda^{\alpha\beta}_{(n)}$$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

1. Harmonic gauge condition is not yet satisfied

$$w^{\alpha}_{(n)} = \partial_\mu u^{\alpha\mu}_{(n)} = \text{FP}\,\Box^{-1}_{\text{ret}}\left[B\,r^{B-1} n_i \Lambda^{\alpha i}_{(n)}\right]$$

2. But $\Box w^{\alpha}_{(n)} = 0$ hence we can compute $v^{\alpha\beta}_{(n)}$ such that at once

$$\Box v^{\alpha\beta}_{(n)} = 0 \quad \text{and} \quad \partial_\mu v^{\alpha\mu}_{(n)} = -w^{\alpha}_{(n)}$$

3. Thus we define

$$\boxed{h^{\alpha\beta}_{(n)} = u^{\alpha\beta}_{(n)} + v^{\alpha\beta}_{(n)}}$$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

Theorem 1:
The MPM solution is the most general solution of Einstein's vacuum equations outside an isolated matter system

Theorem 2:
The general structure of the PN expansion is

$$h^{\alpha\beta}_{\mathsf{PN}}(\mathbf{x},t,c) = \sum_{\substack{p\geqslant 2\\ q\geqslant 0}} \frac{(\ln c)^q}{c^p} h^{\alpha\beta}_{p,q}(\mathbf{x},t)$$

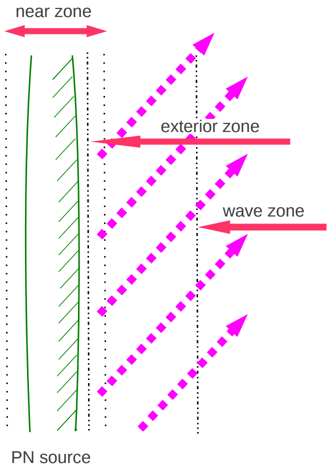
Theorem 3:
The MPM solution is asymptotically simple at future null infinity in the sense of Penrose [1963, 1965] and agrees with the Bondi-Sachs [1962] formalism

$$\underbrace{M_{\mathsf{B}}(u)}_{\text{Bondi mass}} = \underbrace{M}_{\text{ADM mass}} - \frac{G}{5c^5}\int_{-\infty}^{u} \mathrm{d}\tau M^{(3)}_{ij}(\tau) M^{(3)}_{ij}(\tau)$$

$+$ higher multipoles and higher PM computable to any order

The MPM-PN formalism

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



The MPM-PN formalism

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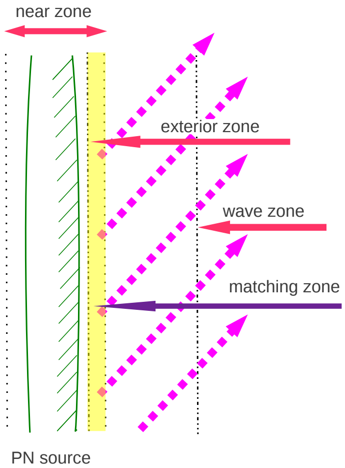


The matching equation

[*e.g.* Lagerström *et al.* 1967; Burke & Thorne 1971; Kates 1980; Anderson *et al.* 1982; Blanchet 1998]

1. This is a variant of the theory of matched asymptotic expansions

$$\text{match} \quad \begin{cases} \text{the multipole expansion} \quad \mathcal{M}(h^{\alpha\beta}) \equiv h^{\alpha\beta}_{\text{MPM}} \\ \qquad\qquad \text{with} \\ \text{the PN expansion} \quad \bar{h}^{\alpha\beta} \equiv h^{\alpha\beta}_{\text{PN}} \end{cases}$$

$$\boxed{\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})}$$

   - Left side is the NZ expansion ($r \to 0$) of the exterior MPM field
   - Right side is the FZ expansion ($r \to +\infty$) of the inner PN field

2. The matching equation has been implemented at any post-Minkowskian order in the exterior field and any PN order in the inner field

3. It gives a unique (formal) multipolar-post-Newtonian solution valid everywhere inside and outside the source

General solution for the multipolar field [Blanchet 1995, 1998]

$$\mathcal{M}(h^{\mu\nu}) = \text{FP}\Box^{-1}_{\text{ret}}\mathcal{M}(\Lambda^{\mu\nu}) + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{M_L^{\mu\nu}(t-r/c)}{r}\right\}}_{\text{homogeneous retarded solution}}$$

where
$$M_L^{\mu\nu}(t) = \text{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\,\underbrace{\bar{\tau}^{\mu\nu}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}$$

- The FP procedure plays the role of an UV regularization in the non-linearity term but an IR regularization in the multipole moments
- From this one obtains the multipole moments of the source at any PN order solving the wave generation problem

General solution for the inner PN field

[Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

$$\bar{h}^{\mu\nu} = \mathrm{FP}\,\Box^{-1}_{\text{ret}}\bar{\tau}^{\mu\nu} + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{R_L^{\mu\nu}(t-r/c)-R_L^{\mu\nu}(t+r/c)}{r}\right\}}_{\text{homogeneous antisymmetric solution}}$$

where $$R_L^{\mu\nu}(t) = \mathrm{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_1^{+\infty}\mathrm{d}z\,\gamma_\ell(z)\,\underbrace{\mathcal{M}(\tau^{\mu\nu})(\mathbf{x},t-zr/c)}_{\text{multipole expansion of the pseudo-tensor}}$$

- The radiation reaction effects starting at 2.5PN order appropriate to an isolated system are determined to any order
- In particular nonlinear radiation reaction effects associated with tails are contained in the second term and start at 4PN order

Radiation reaction potentials to 4PN order

$$V^{\mathrm{reac}}(\mathbf{x},t) = -\overbrace{\frac{G}{5c^5}x^{ij}M^{(5)}_{ij}(t)}^{\text{2.5PN piece}} + \frac{G}{c^7}\left[\frac{1}{189}x^{ijk}\,M^{(7)}_{ijk}(t) - \frac{1}{70}\mathbf{x}^2 x^{ij}\,M^{(7)}_{ij}(t)\right]$$
$$\underbrace{- \frac{4G^2M}{5c^8}\,x^{ij}\int_0^{+\infty}\mathrm{d}\tau\,M^{(7)}_{ij}(t-\tau)\left[\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{12}\right]}_{\text{4PN radiation reaction tail}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

$$V_i^{\mathrm{reac}}(\mathbf{x},t) = \frac{G}{c^5}\left[\frac{1}{21}\hat{x}^{ijk}\,M^{(6)}_{jk}(t) - \frac{4}{45}\,\epsilon_{ijk}\,x^{jl}\,S^{(5)}_{kl}(t)\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

Radiative moments at future null infinity

Correct for the logarithmic deviation of retarded time in harmonic coordinates with respect to the actual null coordinate

$$\underbrace{T - \frac{R}{c}}_{\text{radiative coordinates}} = \underbrace{t - \frac{r}{c}}_{\text{harmonic coordinates}} - \frac{2GM}{c^3} \ln\left(\frac{r}{c\tau_0}\right) + \mathcal{O}\left(\frac{1}{r}\right)$$

Asymptotic waveform is parametrized by radiative moments U_L and V_L [Thorne 1980]

$$h_{ij}^{\mathsf{TT}} = \frac{1}{R} \sum_{\ell=2}^{\infty} N_{L-2} \underbrace{U_{ijL-2}(T - R/c)}_{\text{mass-type}} + \epsilon_{ab(i} N_{aL-1} \underbrace{V_{j)bL-2}(T - R/c)}_{\text{current-type}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

The 4.5PN radiative quadrupole moment

[Marchand, Blanchet & Faye 2016]

$$U_{ij}(t) = M^{(2)}_{ij}(t) + \underbrace{\frac{GM}{c^3}\int_0^{+\infty} d\tau\, M^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}}$$

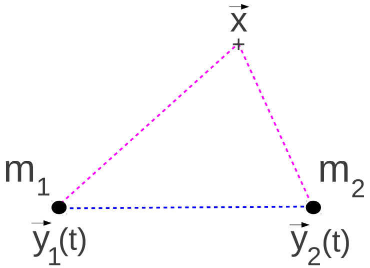
$$+\frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty} d\tau\, M^{(3)}_{a<i}M^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\}$$

$$+\underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty} d\tau\, M^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}}$$

$$+\underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty} d\tau\, M^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}}$$

$$+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

Problem of point particles



$$U(\mathbf{x},t) = \frac{Gm_1}{|\mathbf{x}-\mathbf{y}_1(t)|} + \frac{Gm_2}{|\mathbf{x}-\mathbf{y}_2(t)|}$$

$$\frac{\mathrm{d}^2\mathbf{y}_1}{\mathrm{d}t^2} = (\boldsymbol{\nabla}U)\,(\mathbf{y}_1(t),t) \stackrel{?}{=} -Gm_2\frac{\mathbf{y}_1-\mathbf{y}_2}{|\mathbf{y}_1-\mathbf{y}_2|^3}$$

- For extended bodies the self-acceleration of the body cancels out by Newton's action-reaction law
- For point particles one needs a self-field regularization to remove the infinite self-field of the particle

Dimensional regularization

[t'Hooft & Veltman 1972; Bollini & Giambiagi 1972; Breitenlohner & Maison 1977]

1. Einstein's field equations are solved in d spatial dimensions (with $d \in \mathbb{C}$) with distributional sources. In Newtonian approximation

$$\Delta U = -4\pi \frac{2(d-2)}{d-1} G\rho$$

2. For two point-particles $\rho = m_1 \delta_{(d)}(\mathbf{x} - \mathbf{y}_1) + m_2 \delta_{(d)}(\mathbf{x} - \mathbf{y}_2)$ we get

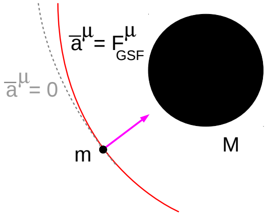
$$U(\mathbf{x}, t) = \frac{2(d-2)k}{d-1} \left(\frac{G m_1}{|\mathbf{x} - \mathbf{y}_1|^{d-2}} + \frac{G m_2}{|\mathbf{x} - \mathbf{y}_2|^{d-2}} \right) \quad \text{with} \quad k = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}$$

3. Computations are performed when $\Re(d)$ is a large negative number, and the result is analytically continued for any $d \in \mathbb{C}$ except for isolated poles
4. Dimensional regularization is then followed by a renormalization of the worldline of the particles so as to absorb the poles $\propto (d-3)^{-1}$

Problem of the gravitational self-force (GSF)

[Mino, Sasaki & Tanaka 1997; Quinn & Wald 1997; Detweiler & Whiting 2003]

- A particle is moving on a background space-time of a massive black hole
- Its stress-energy tensor modifies the background gravitational field
- Because of the back-reaction the motion of the particle deviates from a background geodesic hence the gravitational self force

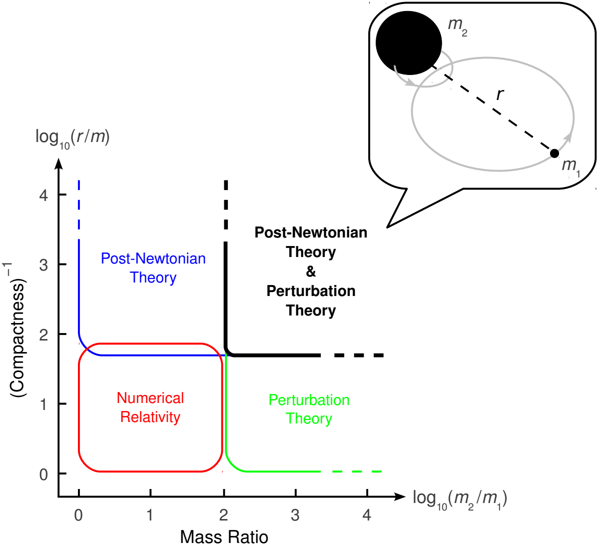


$$\boxed{\bar{a}^\mu = F^\mu_{\mathsf{GSF}} = \mathcal{O}\left(\frac{m}{M}\right)}$$

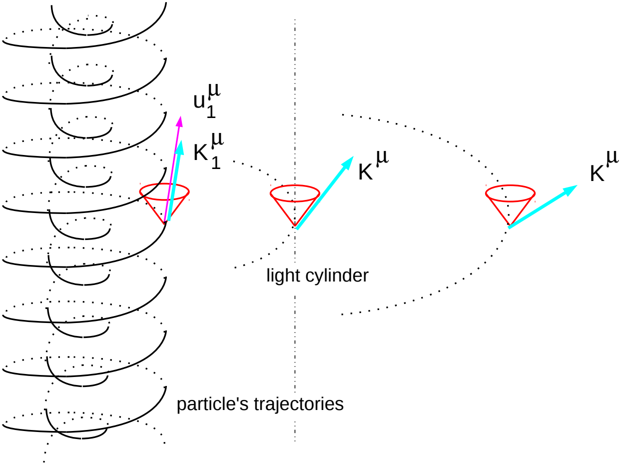
The GSF is computed to high accuracy by

- numerical methods [Sago, Barack & Detweiler 2008; Shah, Friedmann & Whiting 2014]
- analytical ones [Mano, Susuki & Takasugi 1996; Bini & Damour 2013, 2014]

Checking the PN machinery with GSF



Looking at the conservative part of the dynamics



Space-time for exact circular orbits admits a Helical Killing Vector (HKV) K^μ

Choice of a gauge-invariant observable [Detweiler 2008]

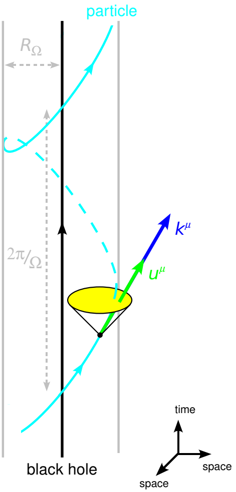
1. For exactly circular orbits the geometry admits a helical Killing vector with

$$K^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi \quad \text{(asymptotically)}$$

2. The four-velocity of the particle is necessarily tangent to the Killing vector hence

$$K_1^\mu = z_1\, u_1^\mu$$

3. This z_1 is the Killing energy of the particle associated with the HKV and is also a redshift
4. The relation $z_1(\Omega)$ is well-defined in both PN and GSF approaches and is gauge-invariant

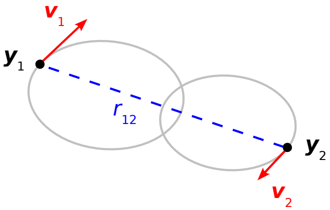


PN calculation of the redshift factor

[Blanchet, Detweiler, Le Tiec & Whiting 2010, 2011; Blanchet, Faye & Whiting 2014, 2015]

1. In a coordinate system such that $K^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi$ everywhere this invariant quantity reduces to the zero-th component of the particle's four-velocity,

$$u_1^t = \frac{1}{z_1} = \Big(- \underbrace{\mathbf{Reg}_1 \left[g_{\mu\nu} \right]}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2} \Big)^{-1/2}$$



2. One needs a self-field regularization
   - Hadamard regularization will yield an ambiguity at 3PN order
   - Dimensional regularization will be free of any ambiguity at 3PN order

Standard PN theory agrees with GSF calculations

$$
\begin{aligned}
u^t_{\mathrm{SF}} &= -y - 2y^2 - 5y^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4 \\
&+ \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_{\mathsf{E}} - \frac{64}{5}\ln(16y)\right) y^5 \\
&- \frac{956}{105} y^6 \ln y - \frac{13696\pi}{525} y^{13/2} - \frac{51256}{567} y^7 \ln y + \frac{81077\pi}{3675} y^{15/2} \\
&+ \frac{27392}{525} y^8 \ln^2 y + \frac{82561159\pi}{467775} y^{17/2} - \frac{27016}{2205} y^9 \ln^2 y \\
&- \frac{11723776\pi}{55125} y^{19/2} \ln y - \frac{4027582708}{9823275} y^{10} \ln^2 y \\
&+ \frac{99186502\pi}{1157625} y^{21/2} \ln y + \frac{23447552}{165375} y^{11} \ln^3 y + \cdots
\end{aligned}
$$

1. Integral PN terms such as 3PN permit checking dimensional regularization
2. Half-integral PN terms starting at 5.5PN order permit checking the non-linear tails (and tail-of-tails)

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1. Integral PN terms such as 3PN permit checking dimensional regularization
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4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\cdots\right]n_{12}^i+\cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

2PN:

[Otha, Okamura, Kimura & Hiida 1973, 1974; Damour & Schäfer 1985]	ADM Hamiltonian
[Damour & Deruelle 1981; Damour 1983]	Harmonic coordinates
[Kopeikin 1985; Grishchuk & Kopeikin 1986]	Extended fluid balls
[Blanchet, Faye & Ponsot 1998]	Direct PN iteration
[Itoh, Futamase & Asada 2001]	Surface integral method

4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\cdots\right]n_{12}^i+\cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\text{2.5PN radiation reaction}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\text{3.5PN radiation reaction}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\text{4PN conservative \& radiation tail}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN

[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
[Foffa & Sturani 2011]	Effective field theory

4PN

[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017ab]	Fokker Lagrangian
[Foffa & Sturani 2012, 2013] (partial results)	Effective field theory

3.5PN energy flux of compact binaries

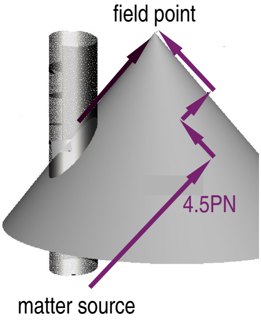
[BDIWW 1995; B 1996, 1998; BFIJ 2002; BDEI 2006]

$$
\begin{aligned}
\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 &+ \overbrace{\left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x}^{\text{1PN}} + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}} \\
&+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \overbrace{\left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2}}^{\text{2.5PN tail}} \\
&+ \Bigg[\frac{6643739519}{69854400} + \overbrace{\frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln(16\,x)}^{\text{3PN tail-of-tail}} \\
&\qquad + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\Bigg] x^3 \\
&+ \underbrace{\left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(\frac{1}{c^8}\right) \Bigg\}
\end{aligned}
$$

4.5PN tail interactions between moments

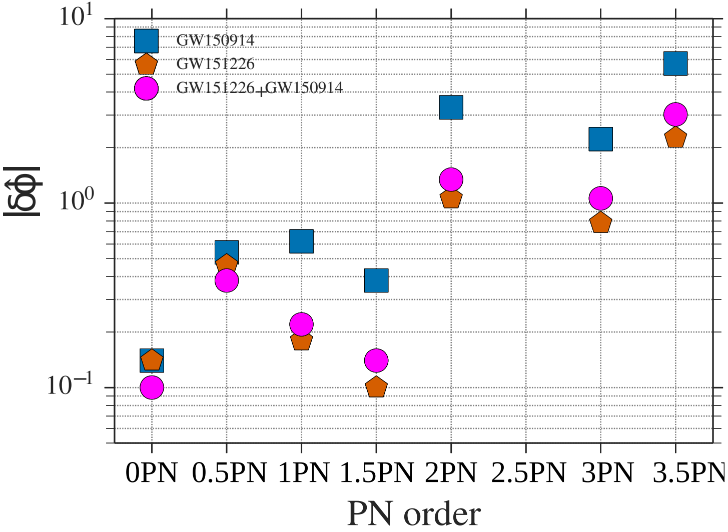
[Marchand, Blanchet & Faye 2016]

$$\mathcal{F}^{4.5\mathrm{PN}} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ \bigg(\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\mathrm{E} - \frac{3424}{105}\ln(16x) + \left[\frac{2062241}{22176} + \frac{41}{12}\pi^2\right]\nu - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3 \bigg) \pi x^{9/2} \Bigg\}$$

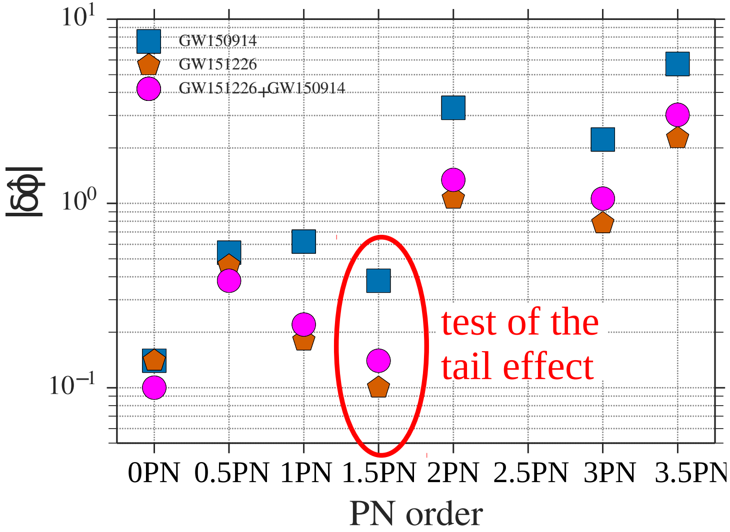


- Perfect agreement with results from BH perturbation theory in the small mass ratio limit $\nu \to 0$ [Tanaka, Tagoshi & Sasaki 1996]
- However the 4PN term in the flux is still in progress

Measurement of PN parameters [LIGO/VIRGO 2016]



Measuring GW tails [Blanchet & Sathyaprakash 1994, 1995]



3.5PN dominant gravitational wave modes

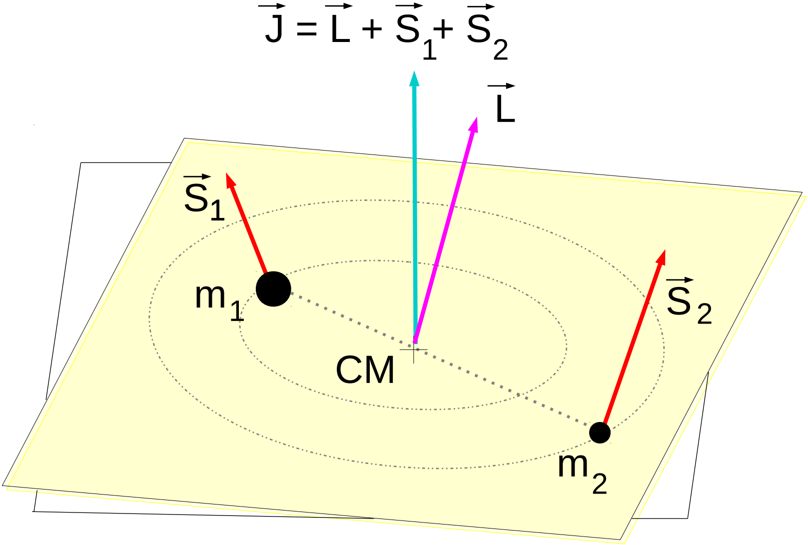
[Faye, Marsat, Blanchet & Iyer 2012; Faye, Blanchet & Iyer 2014]

$$
\begin{aligned}
h_{22} = \frac{2G\,m\,\nu\,x}{R\,c^2}\sqrt{\frac{16\pi}{5}}\,e^{-2\mathrm{i}\psi}\Bigg\{ & 1 + x\left(-\frac{107}{42}+\frac{55\nu}{42}\right) + 2\pi x^{3/2} \\
& + x^2\left(-\frac{2173}{1512}-\frac{1069\nu}{216}+\frac{2047\nu^2}{1512}\right) \\
& + \underbrace{[\cdots]\ x^{5/2}}_{\text{2.5PN}} + \underbrace{[\cdots]\ x^{3}}_{\text{3PN}} + \underbrace{[\cdots]\ x^{7/2}}_{\text{3.5PN}} + \mathcal{O}\left(x^4\right)\Bigg\} \\
h_{33} = \cdots & \\
h_{31} = \cdots &
\end{aligned}
$$

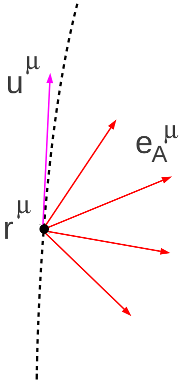
Tail contributions in this expression are factorized out in the phase variable

$$
\psi = \phi - \frac{2GM\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right)
$$

Spinning compact binaries



Spinning particles in a pole-dipole approximation



- Spin degrees of freedom described by an orthonormal moving tetrad along the particle's worldline

$$g_{\mu\nu}\, e_A{}^{\mu} e_B{}^{\nu} = \eta_{AB}$$

- The rotation tensor of the tetrad is defined as

$$\frac{\mathrm{D}e_A{}^{\mu}}{\mathrm{d}\tau} = -\Omega^{\mu\nu} e_{A\nu}$$

- Because of the orthonormality condition the rotation tensor is antisymmetric

$$\Omega^{\mu\nu} = -\Omega^{\nu\mu}$$

- The dynamical degrees of freedom of the particle are the particle's position and the moving tetrad, and the internal structure of the particle is neglected

Action for a system of spinning point particles

[Hanson & Regge 1974; Bailey & Israel 1975]

$$S\left[r^\mu, e_A{}^\mu\right] = \sum_{\text{particles}} \int_{-\infty}^{+\infty} \mathrm{d}\tau \, L\left(u^\mu, \Omega^{\mu\nu}, g_{\mu\nu}\right)$$

The particle's linear momentum and spin tensor are defined by

$$p_\mu = \frac{\partial L}{\partial u^\mu} \qquad S_{\mu\nu} = 2\frac{\partial L}{\partial \Omega^{\mu\nu}}$$

The action should be

1. Lorentz scalar
2. covariant scalar

$$2\frac{\partial L}{\partial g_{\mu\nu}} = p^\mu u^\nu + S^\mu{}_\rho \Omega^{\nu\rho}$$

3. invariant by worldline reparametrization $\tau \to \lambda\tau$

$$L = p_\mu u^\mu + \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu}$$

Equations of motion and spin precession

1. Varying the action with respect to the tetrad ${e_A}^\mu$ (holding the metric $g_{\mu\nu}$ fixed) gives the spin precession equation

$$\boxed{\frac{\mathrm{D}S_{\mu\nu}}{\mathrm{d}\tau} = p_\mu u_\nu - p_\nu u_\mu}$$

2. Varying the action with respect to the position r^μ gives the famous Mathisson-Papapetrou [1937; 1951] equation of motion

$$\boxed{\frac{\mathrm{D}p_\mu}{\mathrm{d}\tau} = -\frac{1}{2} u^\nu R_{\mu\nu\rho\sigma} S^{\rho\sigma}}$$

3. Varying the action with respect to the metric $g_{\mu\nu}$ (keeping $e_{A[\mu}\delta {e^A}_{\nu]} = 0$) gives the stress-energy tensor of the spinning particles [Trautman 1958; Dixon 1979]

$$\boxed{T^{\mu\nu} = \sum_{\text{particles}} \int \mathrm{d}\tau\, p^{(\mu}\, u^{\nu)}\, \frac{\delta^{(4)}(x-r)}{\sqrt{-g}} - \nabla_\rho \int \mathrm{d}\tau\, S^{\rho(\mu}\, u^{\nu)}\, \frac{\delta^{(4)}(x-r)}{\sqrt{-g}}}$$

Spin suplementary condition (SSC)

- To correctly account for the number of degrees of freedom associated with the spin we impose a suplementary condition [Tulczyjew 1957, 1959]

$$\boxed{S^{\mu\nu} p_\nu = 0}$$

- With the latter choice for the SSC, the particle's mass $m^2 = -g^{\mu\nu} p_\mu p_\nu$ and the four-dimensional spin magnitude $s^2 = S^{\mu\nu} S_{\mu\nu}$ are constant

$$\frac{\mathrm{D}m}{\mathrm{d}\tau} = 0 \qquad \frac{\mathrm{D}s}{\mathrm{d}\tau} = 0$$

- The link between the four velocity u^μ and the four linear momentum p^μ is entirely specified, hence the Lagrangian is specified. At linear order in spins

$$p^\mu = m\, u^\mu + \mathcal{O}(S^2)$$

- The equation for the spin reduces to the equation of parallel transport

$$\boxed{\frac{\mathrm{D}S_{\mu\nu}}{\mathrm{d}\tau} = \mathcal{O}(S^2)}$$

4PN spin-orbit effects in the orbital frequency

[Marsat, Bohé, Faye, Blanchet & Buonanno 2013]

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\,\nu\,x^{5/2}\Bigg\{\overbrace{1 + x\,[\cdots] + x^{3/2}\,[\cdots] + x^2\,[\cdots] + x^{5/2}\,[\cdots] + x^3\,[\cdots]}^{\text{non-spin terms}}$$
$$+ \underbrace{[\cdots]\,x^{3/2}}_{\text{1.5PN SO}} + \underbrace{[\cdots]\,x^{2}}_{\text{2PN SS}} + \underbrace{[\cdots]\,x^{5/2}}_{\text{2.5PN SO}} + \underbrace{[\cdots]\,x^{3}}_{\text{3PN SO}_{\text{tail}}\ \&\ \text{SS}}$$
$$+ \underbrace{[\cdots]\,x^{7/2}}_{\text{3.5PN SO}} + \underbrace{[\cdots]\,x^{4}}_{\text{4PN S0}_{\text{tail}}\ \&\ \text{SS}} + \mathcal{O}\left(x^4\right)\Bigg\}$$

- Leading SO and SS terms due to [Kidder, Will & Wiseman 1993; Kidder 1995]
- Many next-to-leading (NL) SS terms mostly in the EOM computed within the ADM Hamiltonian and the Effective Field Theory

Summary of known PN orders

Method	Equations of motion	Energy flux	Waveform
Multipolar-post-Minkowskian & post-Newtonian (MPM-PN)	4PN non-spin 3.5PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN non-spin 4PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN non-spin 1.5PN (L) SO 2PN (L) SS
Canonical ADM Hamiltonian	4PN non-spin 3.5PN (NNL) SO 4PN (NNL) SS 3.5PN (NL) SSS		
Effective Field Theory (EFT)	3PN non-spin 2.5PN (NL) SO 4PN (NNL) SS	2PN non-spin 3PN (NL) SS	
Direct Integration of Relaxed Equations (DIRE)	2.5PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS
Surface Integral	3PN non-spin		

- The 4.5PN non-spin coefficient in the energy flux also known
- Many works devoted to spins:
    - Spin effects (SO, SS, SSS) are known in EOM up to 4PN order
    - SO effects are known in radiation field up to 4PN
    - SS in radiation field known to 3PN