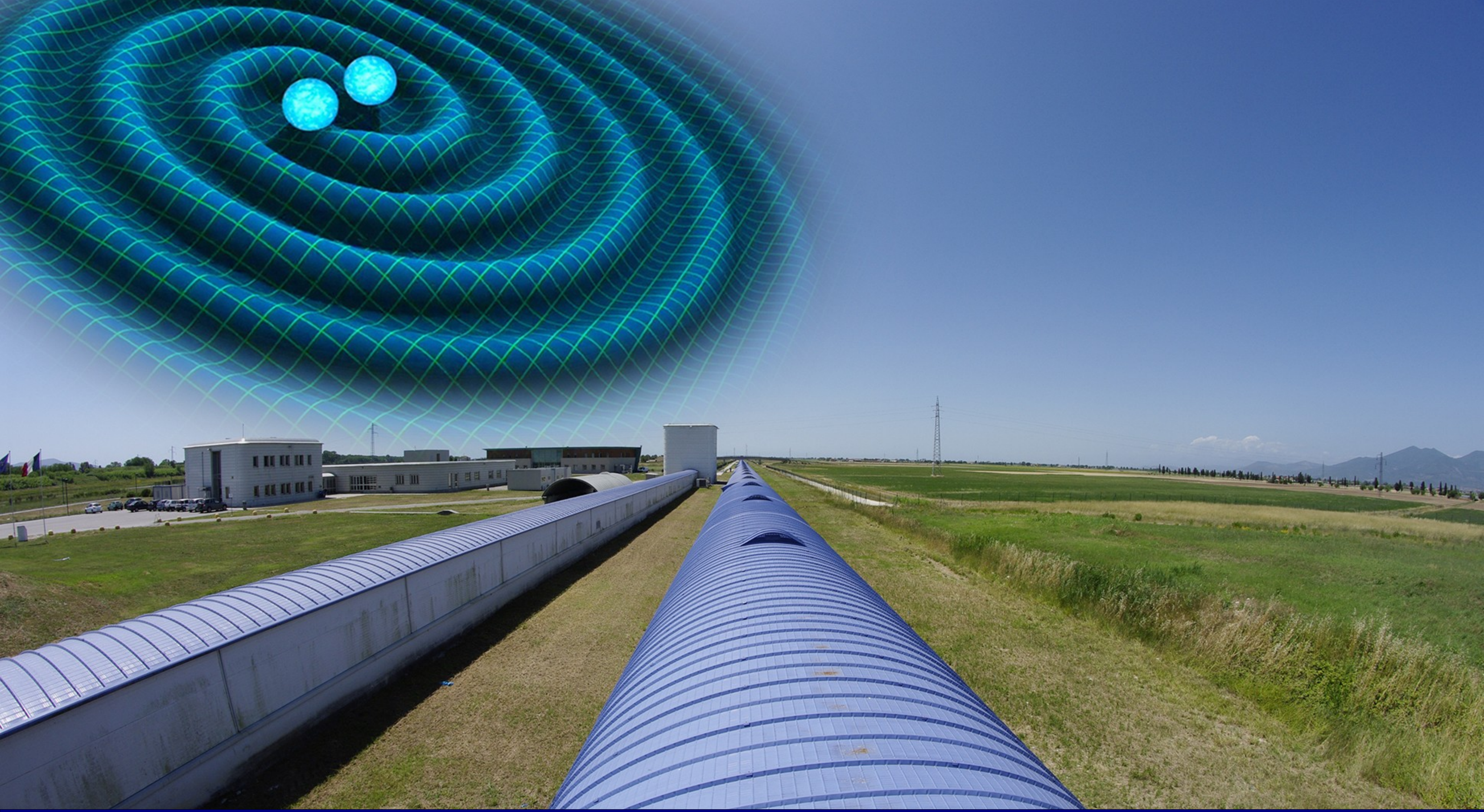
Un trés bon cours en Francais peut se trouver ici:
http://lappweb.in2p3.fr/~buskulic/cours/Notes_cours_Buskulic_Jijel.pdf

Un livre en anglais, plutôt ancien mais toujours essentiel :
Fundamentals of Interferometric Gravitational Wave Detectors
P. Saulson



Gravitational wave detection on Earth (the optical principle)

Jerome Degallaix

École de Gif 2017

For the next 75 minutes

I. Origin and effect of GW

II. Response of a laser interferometer

III. The sensitivity of the Michelson

IV. The upgrade path

I

Origin and effect of GW

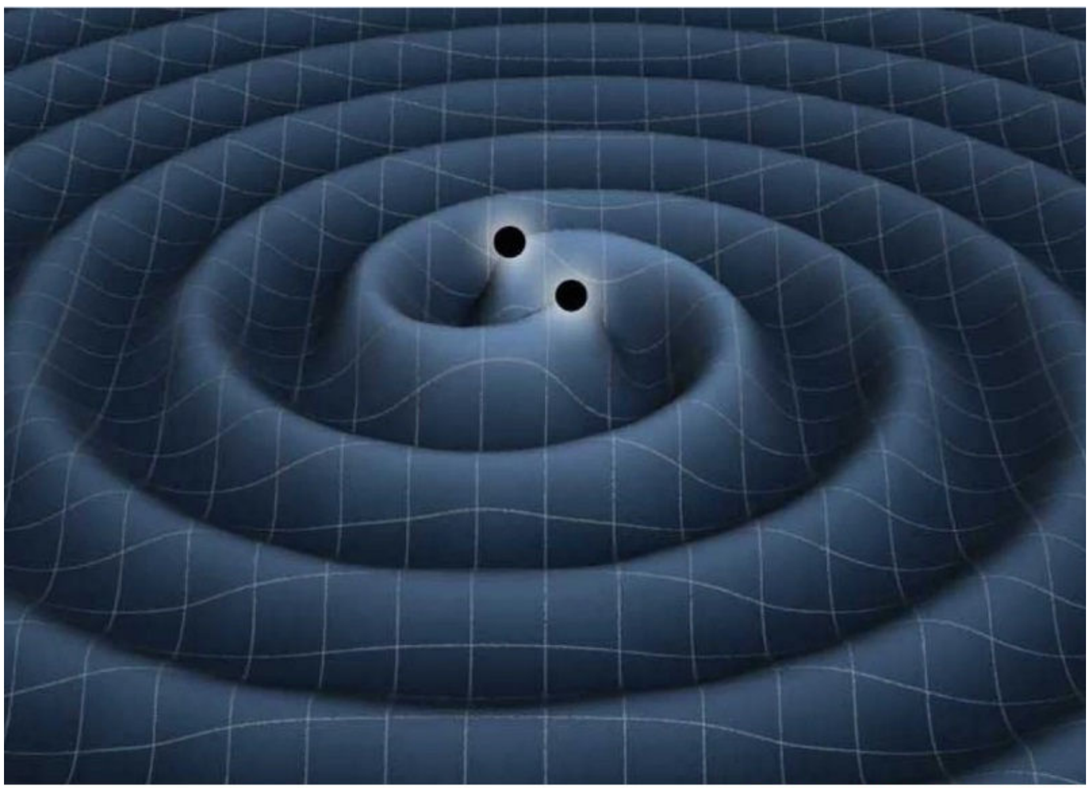
1. a simplified look
2. a more rigorous approach

Gravitational waves

Gravitational waves

=

Perturbations of space time due to acceleration of masses



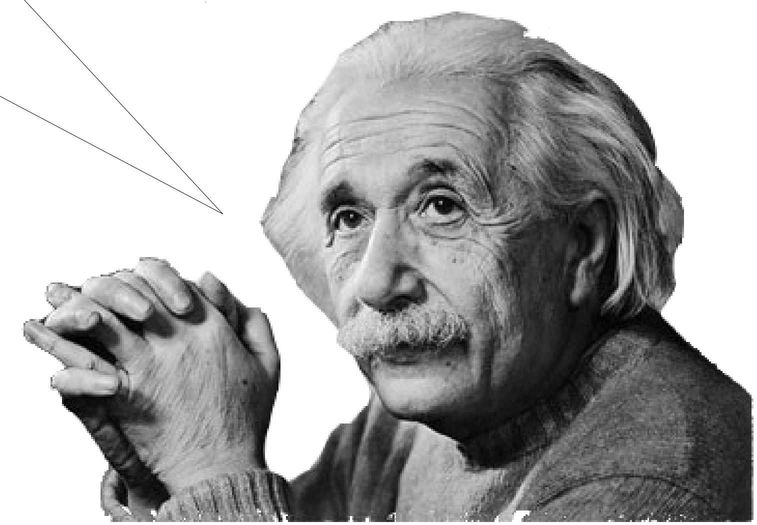
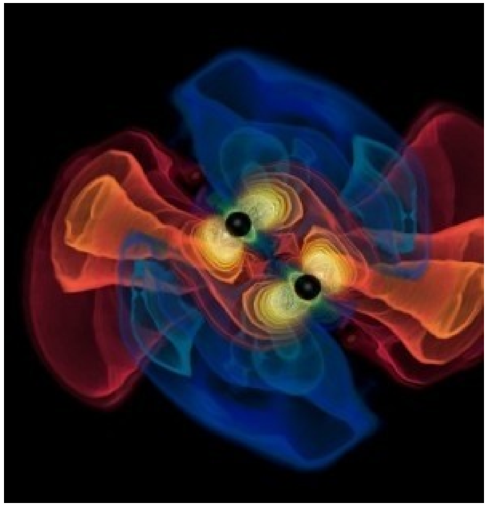


Image credit: NASA / GSFC / Washington Post

Space time ripples

Animation from GW150914 press conference here

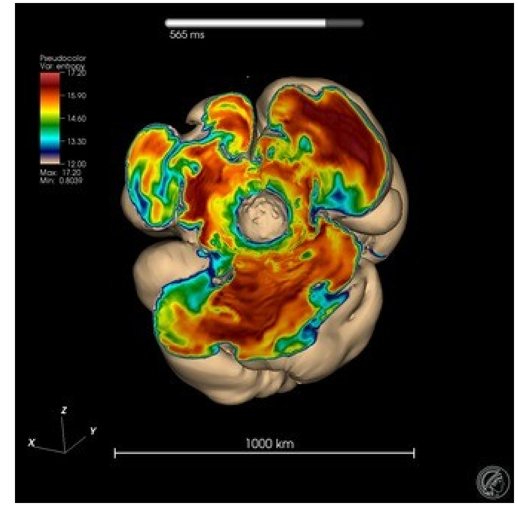
Potential sources



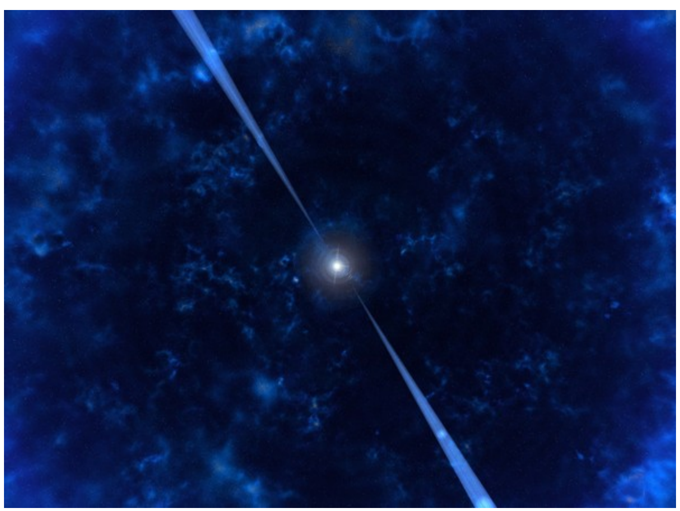
Fusion of compact objets

Short signal

(< 10 s)

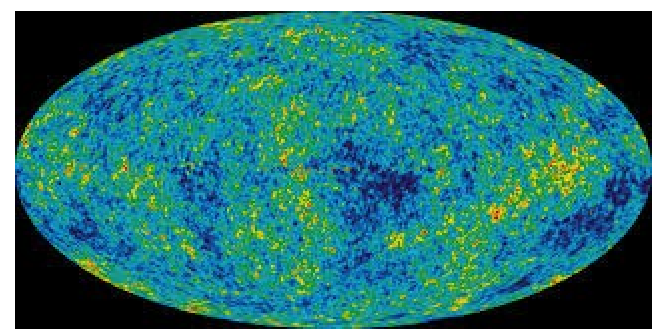


Star core collapse (supernovae)



Pulsars

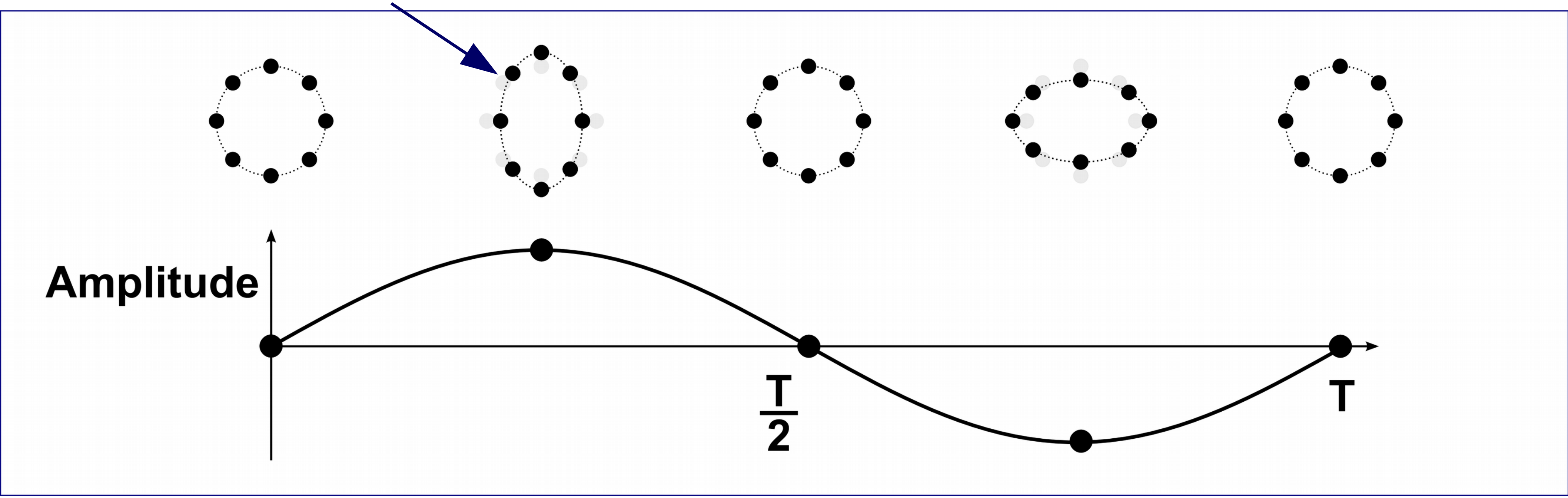
Continious signal



Cosmological background

Effect of gravitational waves

Free falling masses



Gravitational wave amplitude

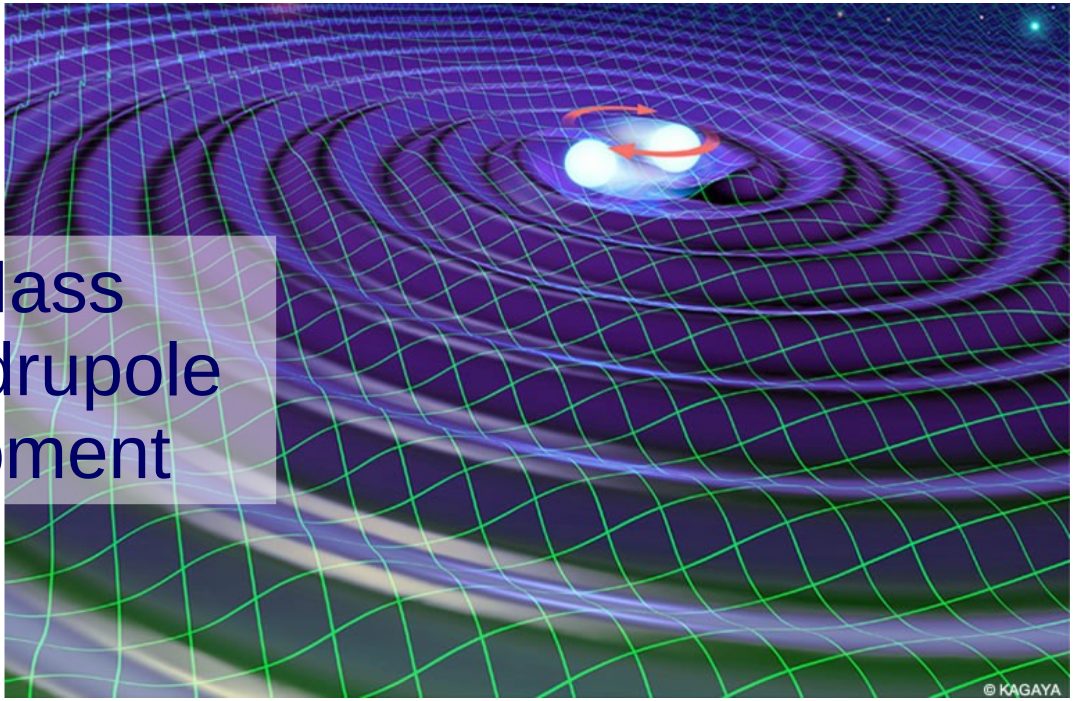
Quadrupole formula :

$$h = \frac{2G}{c^4 r}\ddot{I}$$

Amplitude

Distance

Mass quadrupole moment

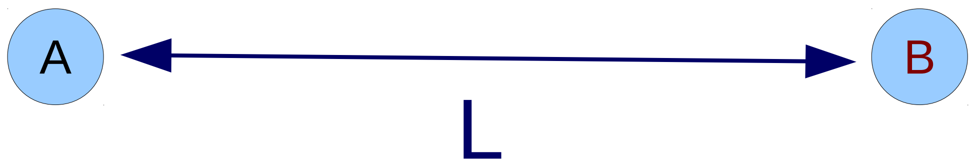


Typical amplitude for two black holes

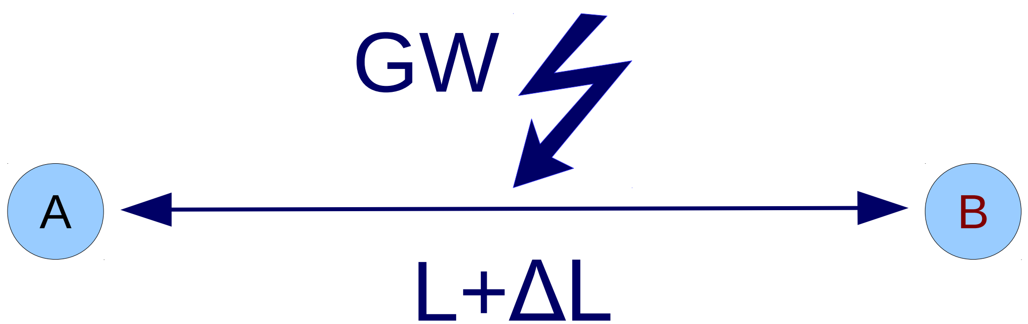
$$h = 1,5 \times 10^{-21} \left(\frac{\text{Mass}}{30\ M_\odot}\right)\left(\frac{400\ \text{Mpc}}{\text{Distance}}\right)\left(\frac{\text{Frequency GW}}{50\ \text{Hz}}\right)^{\frac{2}{3}}$$

Freq. GW = 2 × orbital frequency

Gravitational wave amplitude



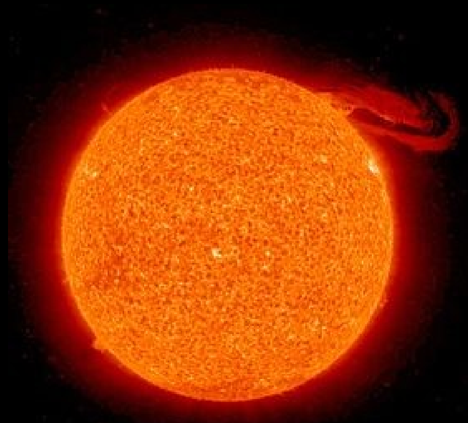
2 free falling masses



Amplitude of the deformation :
$\Delta L = (1/2)\, h \times L$

Order of magnitude $\Delta L = 0.5\, h \times L$

If $h \sim 10^{-21}$ so we should measure :



Sun – Proxima Centauri distance with an accuracy of 0.02 mm

Or 2 km with an accuracy of 10^{-18} meter !

I

Origin and effect of GW

1. a simplify look
2. a more rigorous approach

It all starts from the Einstein equations!

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

Space time curvature Mass-energy distribution

We will make the following assumption :

1. Far from any sources : $T_{\mu\nu} = 0$

2. Small distortions from the flat metric : $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

with : $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $h_{\mu\nu} \ll 1$

And also

3. Using the trace reverse metric : $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$
4. Taking the Lorentz gauge

We finally arrived at the following wave equation :

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t}\right)\bar{h}_{\mu\nu} = 0$$

Using the transverse traceless gauge ($\bar{h}_{\mu\nu} = h_{\mu\nu}$),
a GW propagating along the z axis can be written :

$$\bar{h}_{\mu\nu} = A_{\mu\nu}e^{ik(ct-z)} \qquad A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Calculating the flight path of a photon

Assuming a GW propagating along the z axis (polarisation +) and a photon along the x axis,

The proper time for the photon is :

$$ds^2 = -(cdt)^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + dz^2 = 0$$

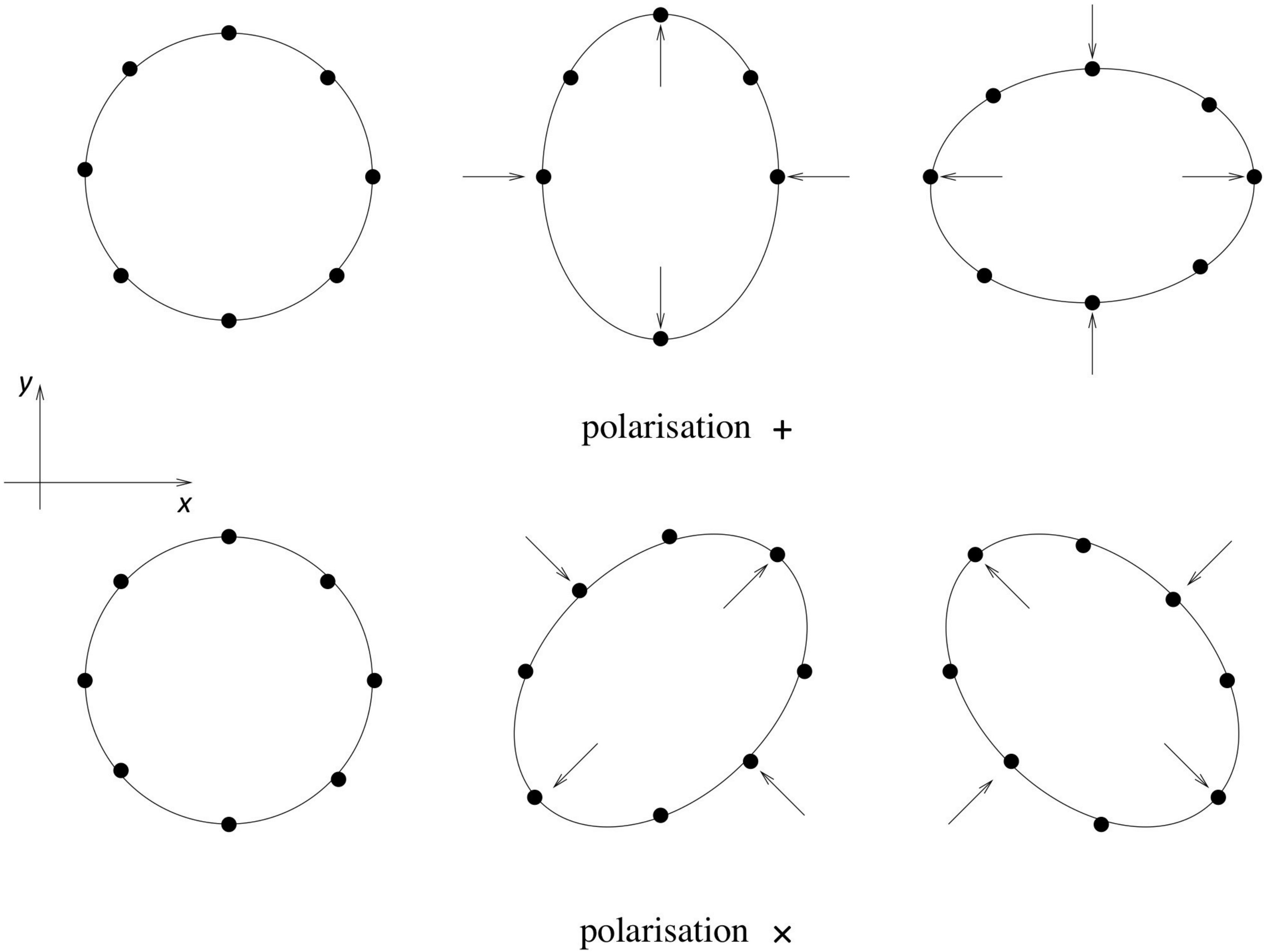
Which becomes :

$$cdt = \left(1 + \frac{1}{2}h_+\right) dx$$

Analog to the previous : $\Delta L = (1/2)\, h \times L$[1]

[1] assuming a uniform plane wave

Effect of the two polarisations



Taken from: http://gravity.paris/en/index.php/2016/02/10/quest-ce-quune-onde-gravitationnelle/

II

Response of a laser interferometer

1. Interaction GW signal - light
2. Response of a Michelson
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4. A more complex configuration

Time modulation due to GW

Starting from the previous equation :

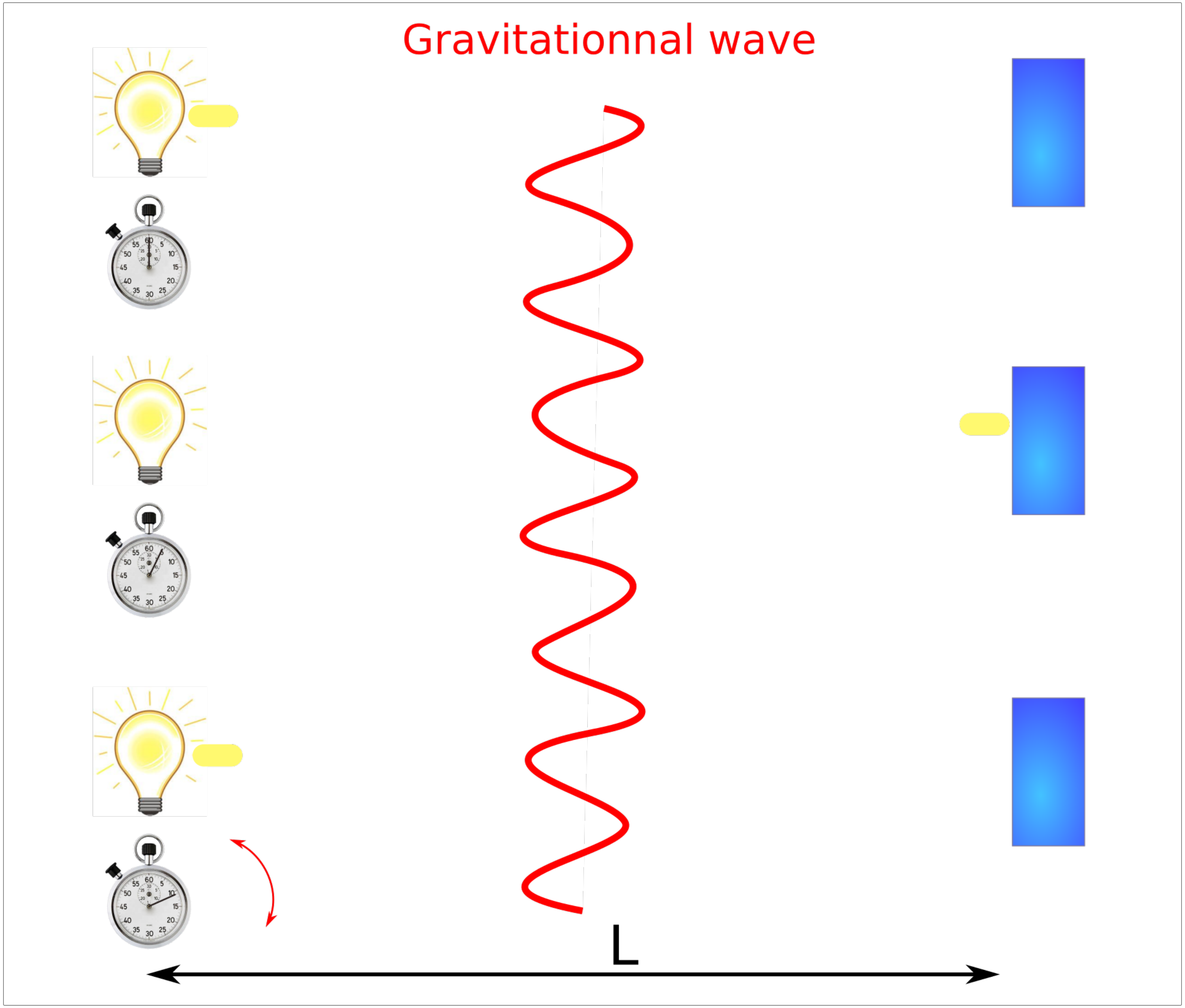
$$\mathrm{cdt} = \left(1 + \frac{1}{2}\mathrm{h}_+\right)\mathrm{dx}$$

To be equivalent to :

$$\mathrm{dx} = \mathrm{c}\left(1 - \frac{1}{2}\mathrm{h}_+\right)\mathrm{dt}$$

We want to calculate the time required for a light beam to do a round trip

Time modulation due to GW



Time modulation due to GW

$$\int_0^{2L} dx = \int_{t_r}^{t} c\left(1 - \frac{1}{2}h(u)\right) du$$

t arrival time, t_r time of departure

$$(t - t_r) = \frac{2L}{c} + \frac{1}{2}\int_{t_r}^{t} h(u)du$$

We suppose a monochromatic GW wave :

$$h(t) = h_0 \cos(\omega_{GW} t)$$

After some algebra (and one assumption) :

$$(t - t_r) = \frac{2L}{c} + \frac{h_0}{\omega_{GW}} \sin\left(\frac{\omega_{GW} L}{c}\right) \cos\left(\omega_{GW}\left(t - L/c\right)\right)$$

Time modulation due to GW

$$(t - t_r) = \frac{2L}{c} + \frac{h_0}{\omega_{GW}} \sin\left(\frac{\omega_{GW} L}{c}\right) \cos\left(\omega_{GW}\left(t - L/c\right)\right)$$

Usual round trip time

Amplitude of the modulation

frequency of the modulation

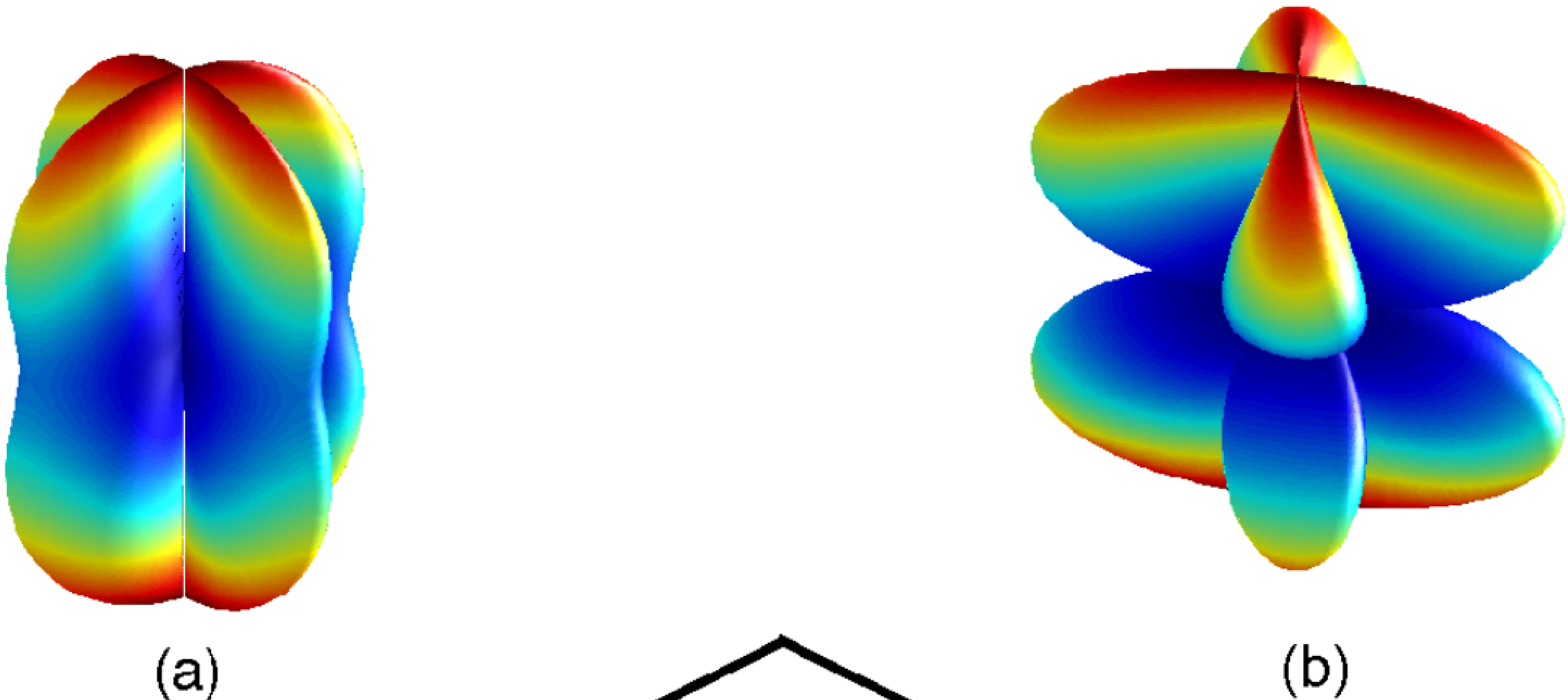
1. For low frequency, we found the usual formula
2. The modulation sign is reversed for the other transverse direction (with + polarisation)
3. No effect for certain GW frequencies
4. Equivalent to a (small) modulated phase shift for the light

Goal: detecting a differential phase shift

With a Michelson interferometer!

Animation from a Michelson here

The detector antenna pattern



(a) + polarisation

(b) × polarisation

You could have some blind post !

II

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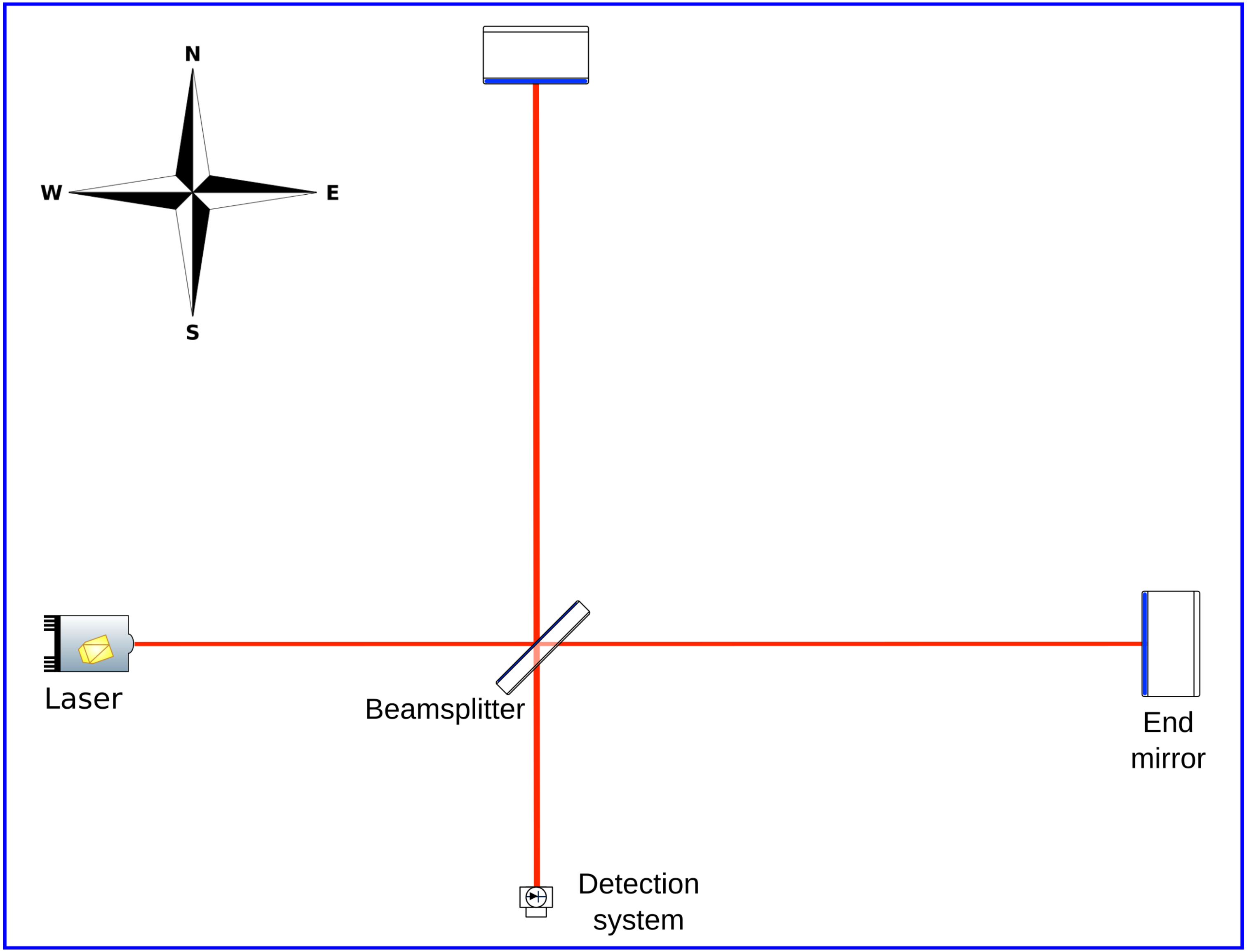
The Michelson interferometer



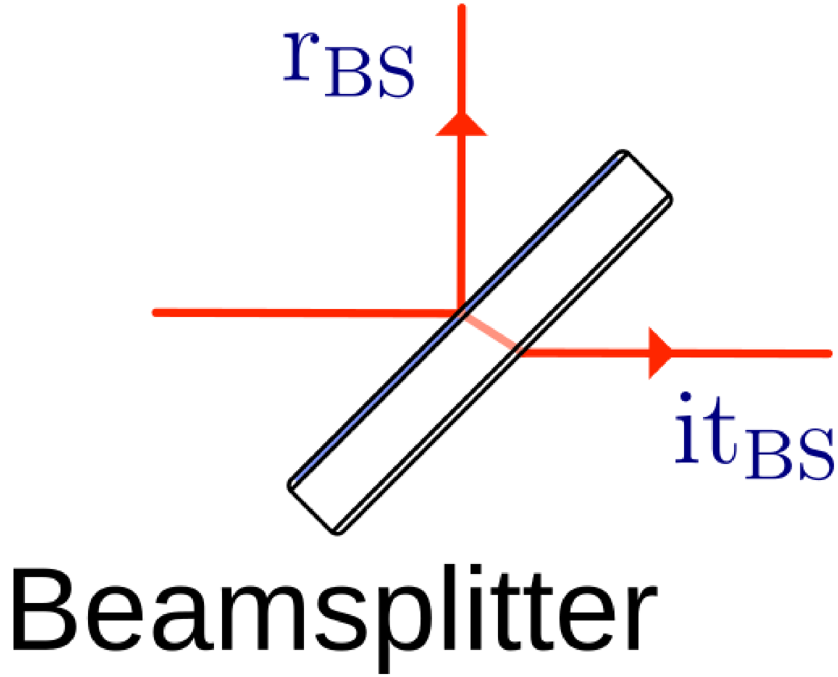
2 arms along the North and East direction (N and E index)

Propagating the electric field

Starting field : E_0

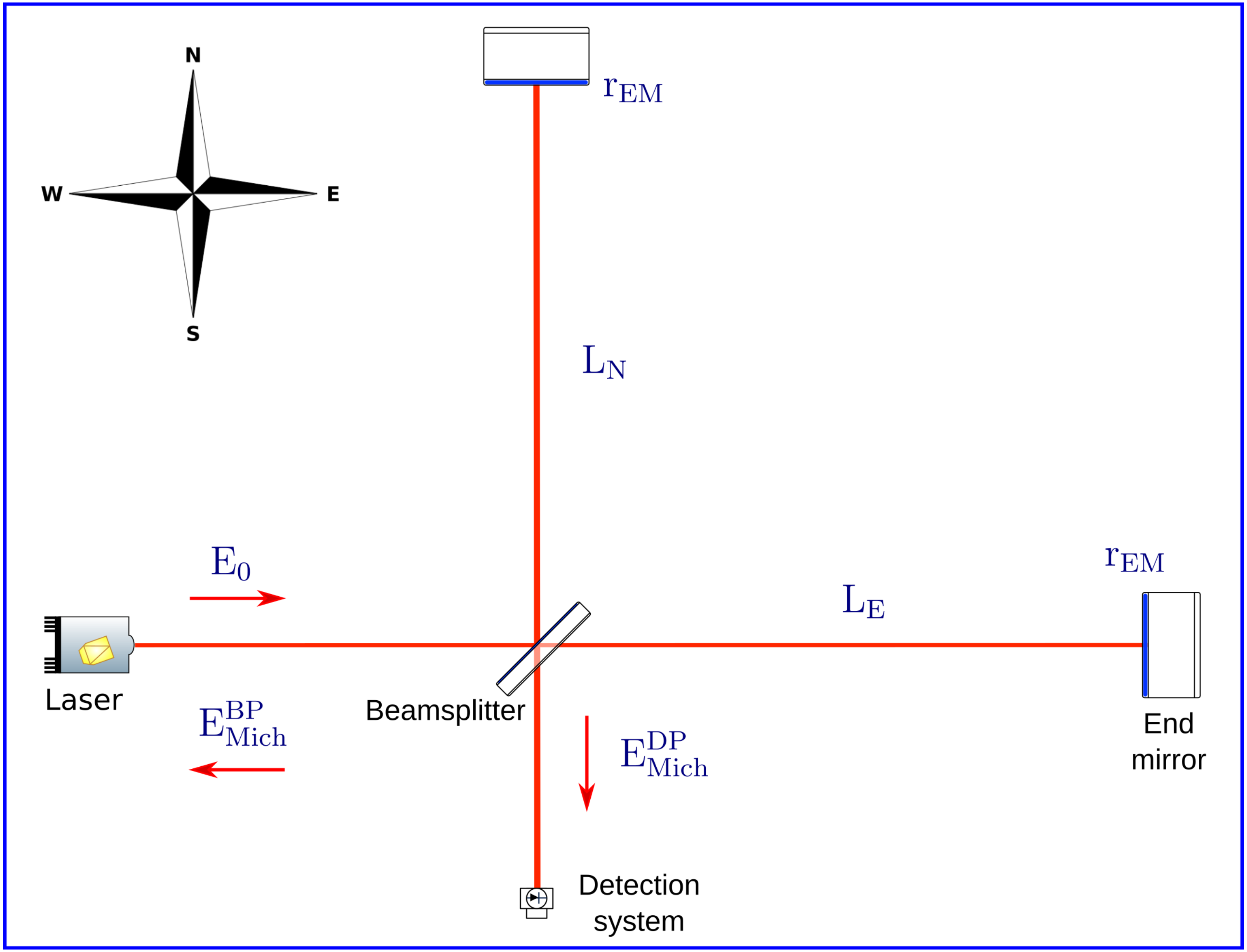
After propagating along a distance L : $\mathrm{E}_1 = \mathrm{e}^{-\mathrm{ikL}} \mathrm{E}_0$

Dealing with the beam splitter :



$$\mathrm{r}_{\mathrm{BS}} = \mathrm{t}_{\mathrm{BS}} = \frac{1}{\sqrt{2}}$$

Convention name for the electric fields



Field equations

$$E^{BP}_{Mich} = \left(r^2_{BS} r_{EM} e^{-i2kL_N} - t^2_{BS} r_{EM} e^{-i2kL_E}\right) E_0$$

$$E^{DP}_{Mich} = \left(i r_{BS} t_{BS} r_{EM} e^{-i2kL_N} + i r_{BS} t_{BS} r_{EM} e^{-i2kL_E}\right) E_0$$

Introducing differential and common lengths for the arms :

$$L_- = \frac{L_N - L_E}{2} \qquad L_+ = \frac{L_N + L_E}{2}$$

$$\Longleftrightarrow$$

$$L_N = L_+ + L_- \qquad L_E = L_+ - L_-$$

Finally, we arrived at :

$$E^{BP}_{Mich} = \left(-ie^{-2kL+} \sin(2kL-)\right) r_{EM} E_0$$

$$E^{DP}_{Mich} = \left(-ie^{-2kL+} \cos(2kL-)\right) r_{EM} E_0$$

Field equations

$$\mathrm{E}^{\mathrm{BP}}_{\mathrm{Mich}} = \left(-\mathrm{i}e^{-2\mathrm{kL}+}\sin(2\mathrm{kL}-)\right)\mathrm{r}_{\mathrm{EM}}\mathrm{E}_0$$

$$\mathrm{E}^{\mathrm{DP}}_{\mathrm{Mich}} = \left(-\mathrm{i}e^{-2\mathrm{kL}+}\cos(2\mathrm{kL}-)\right)\mathrm{r}_{\mathrm{EM}}\mathrm{E}_0$$

From the two previous equations :

1. Energy is preserved between the 2 ports
2. Common motion induces only a phase shift
3. Differential motion modulates the power

The differential phase between the 2 arms due to the GW signal is converted to a variation of power at the dark port.

Increase the phase difference to increase the signal !

Finding the right operating point

Adding a differential modulation due to the passing GW

$$\Delta L_- = \frac{1}{2}\left(L_N\left(1+\frac{h_0 L_N}{2}\cos\left(\omega_{GW}t\right)\right) - L_E\left(1-\frac{h_0 L_E}{2}\cos\left(\omega_{GW}t\right)\right)\right)$$

$$\Delta L_- = L_- + h_0 L_+ \cos\left(\omega_{GW}t\right)$$

Since the amplitude of the GW is very small :

$$\cos(a + x\cos b) \simeq \cos(a) - x\sin(a)\cos(b)$$

$$\sin(a + x\cos b) \simeq \sin(a) - x\cos(a)\cos(b)$$

$$E^{BP}_{Mich} \simeq \left(-ie^{-2kL+}\left(\sin(2kL-) + 2kh_0L_+\cos(2kL_-)\cos\left(\omega_{GW}t\right)\right)\right) r_{EM}E_0$$

$$E^{DP}_{Mich} \simeq \left(ie^{-2kL+}\left(\cos(2kL-) - 2kh_0L_+\sin(2kL_-)\cos\left(\omega_{GW}t\right)\right)\right) r_{EM}E_0$$

Need to be on the dark fringe to maximise the signal on the south port !

Finding the right operating point

But, I do not measure an amplitude but a power with my photodiode...

$$\left|E_{Mich}^{DP}\right|^2 \quad \alpha \quad \left|\cos(2kL_-) - 2kh_0L_+ \sin(2kL_-) \cos\left(\omega_{GW} t\right)\right|^2$$

$$\alpha \quad \cos^2(2kL_-) - 4kh_0L_+ \cos(2kL_-) \sin(2kL_-) \cos\left(\omega_{GW} t\right) + \mathcal{O}(h_0^2)$$

If perfectly on the dark fringe, signal proportional to h_0^2 ,

Need to add a slight dark fringe offset to have a signal proportional to h_0

A closer look at the differential phase

Signal proportional to : $\mathrm{kh_0L_+}$

For a simple Michelson, to increase the detectable signal :

1. Lower the wavelength
2. Increase the length of the arm

Wavelength depends on laser availability and optics, it is fixed at 1064 nm for current interferometers.

Some typical arm lengths

Type of experiments	Arm length
Optomechanics	~1 mm
Large table top experiments	~ 1m
GW prototypes	~ 10 m
Current GW detectors	~ 1 km
Next generation GW detectors	~ 10 km

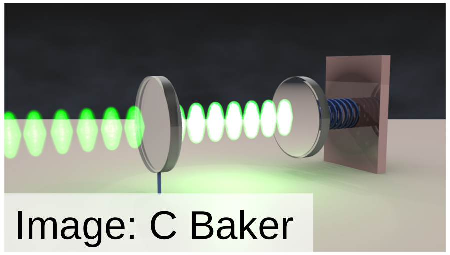
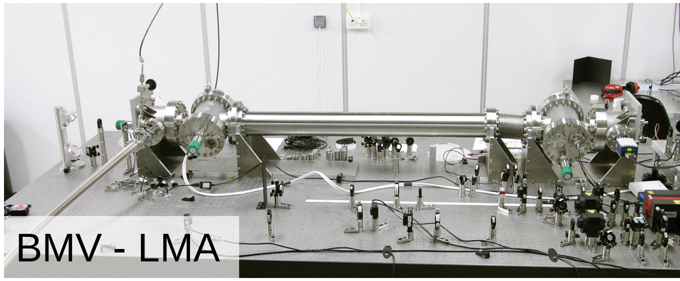
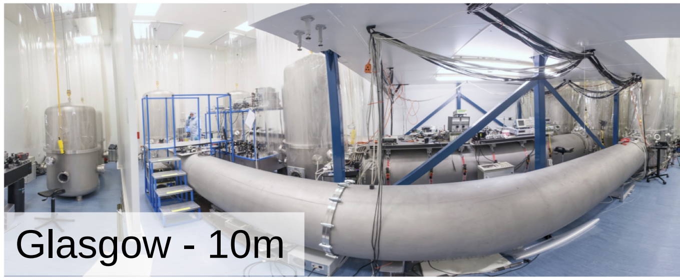


Image: C Baker

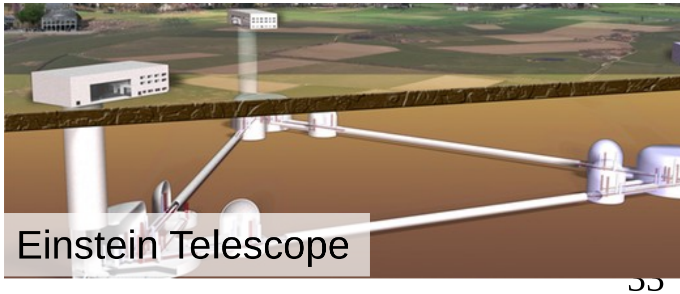


BMV - LMA



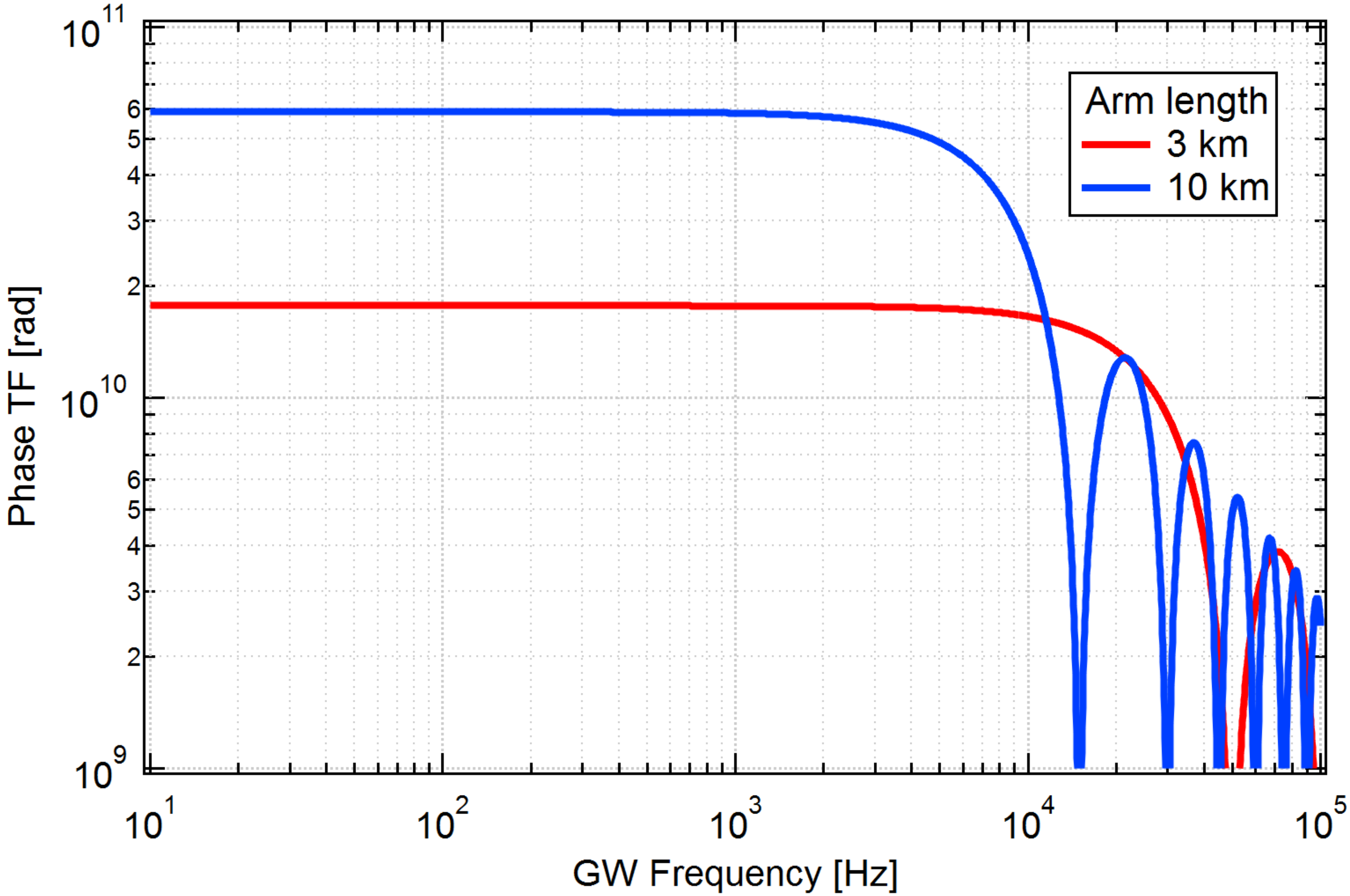
Glasgow - 10m

Virgo



Einstein Telescope

The phase transfer function



Reminder : λ = 1064 nm

$$(t - t_r) = \frac{2L}{c} + \frac{h_0}{\omega_{GW}} \sin\left(\frac{\omega_{GW} L}{c}\right) \cos\left(\omega_{GW}\left(t - L/c\right)\right)$$

II

Response of a laser interferometer

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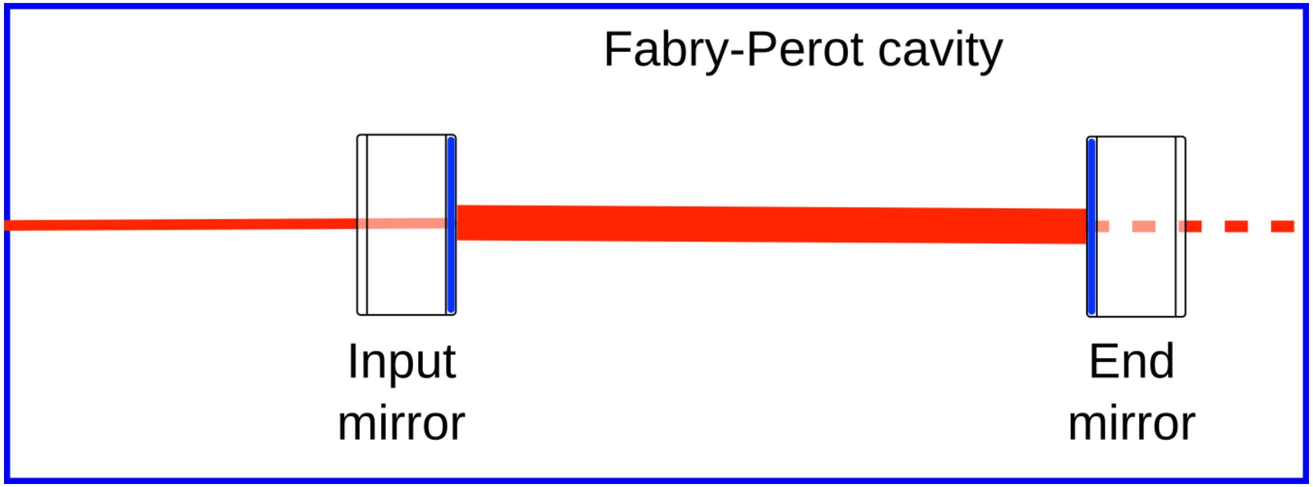
4. A more complex configuration

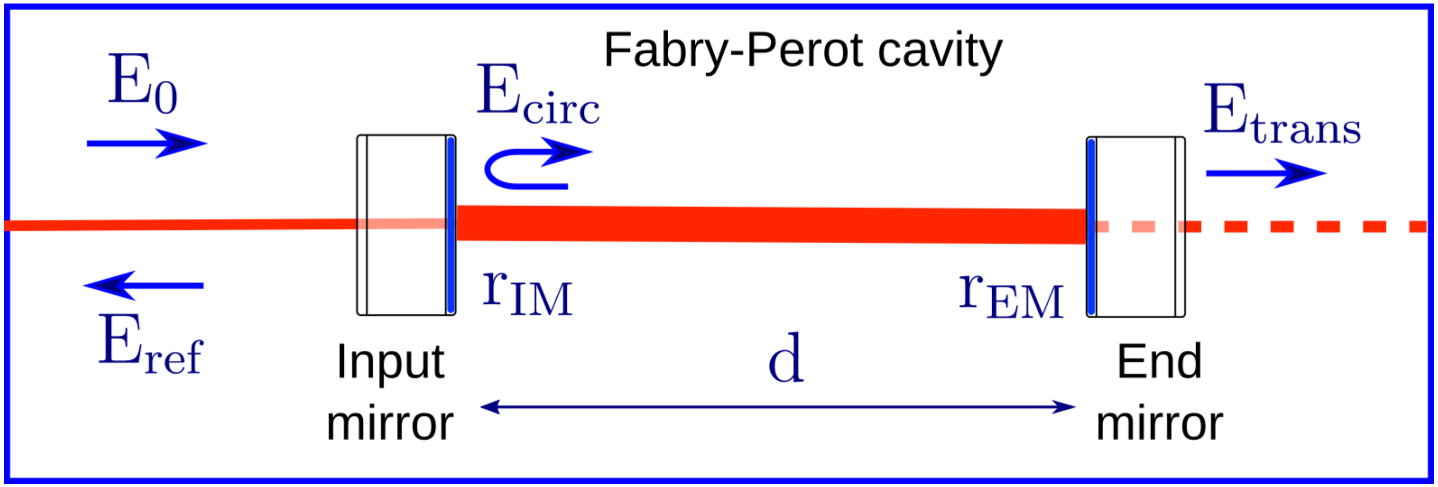
The Fabry-Perot cavity

Two mirrors facing each other separated by a certain distance.



Presence of light interferences inside the cavity, enhancing or destroying the electric field between the 2 mirrors.

Cavity fields



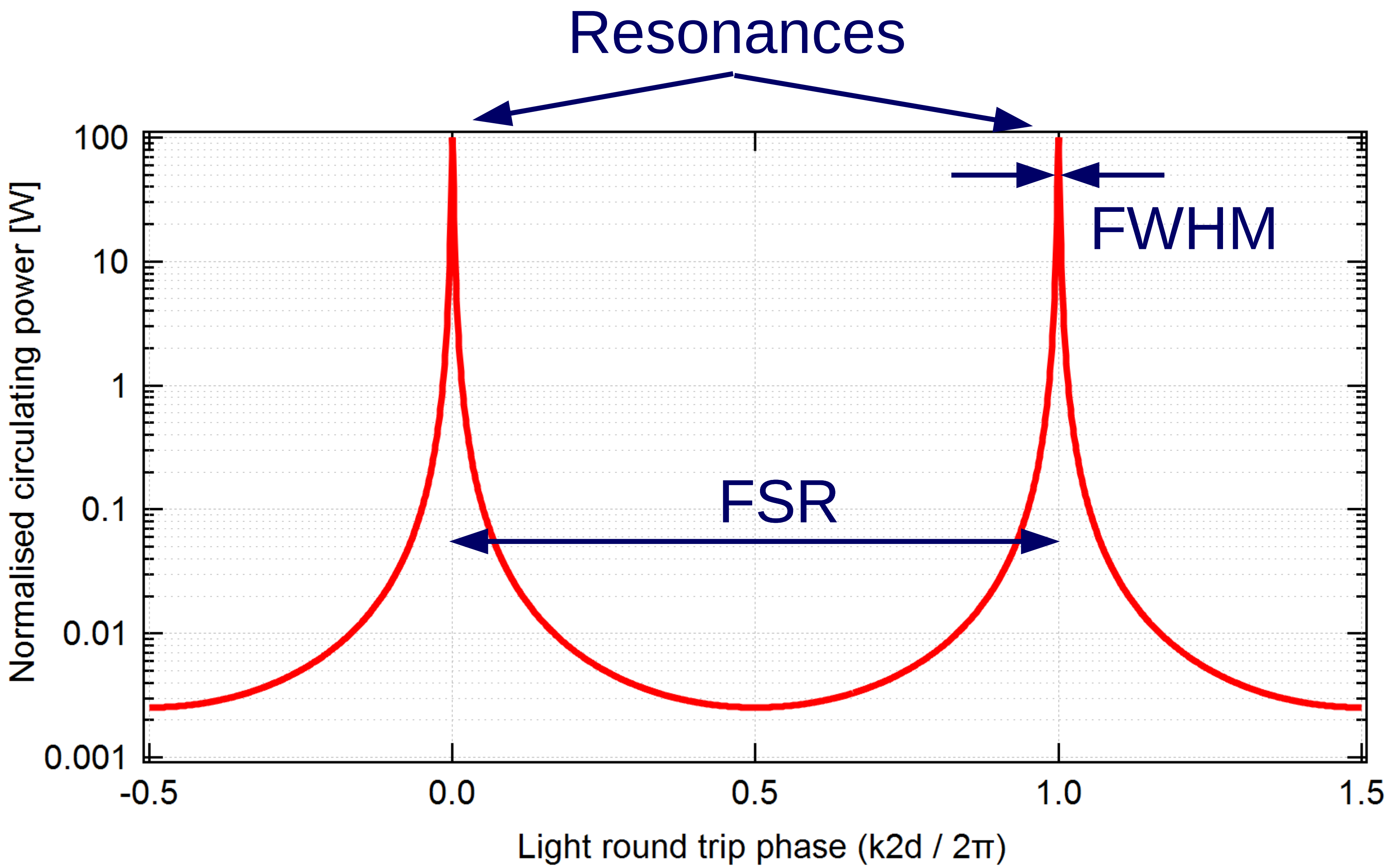
$$r_{xx}^2 + t_{xx}^2 = 1$$

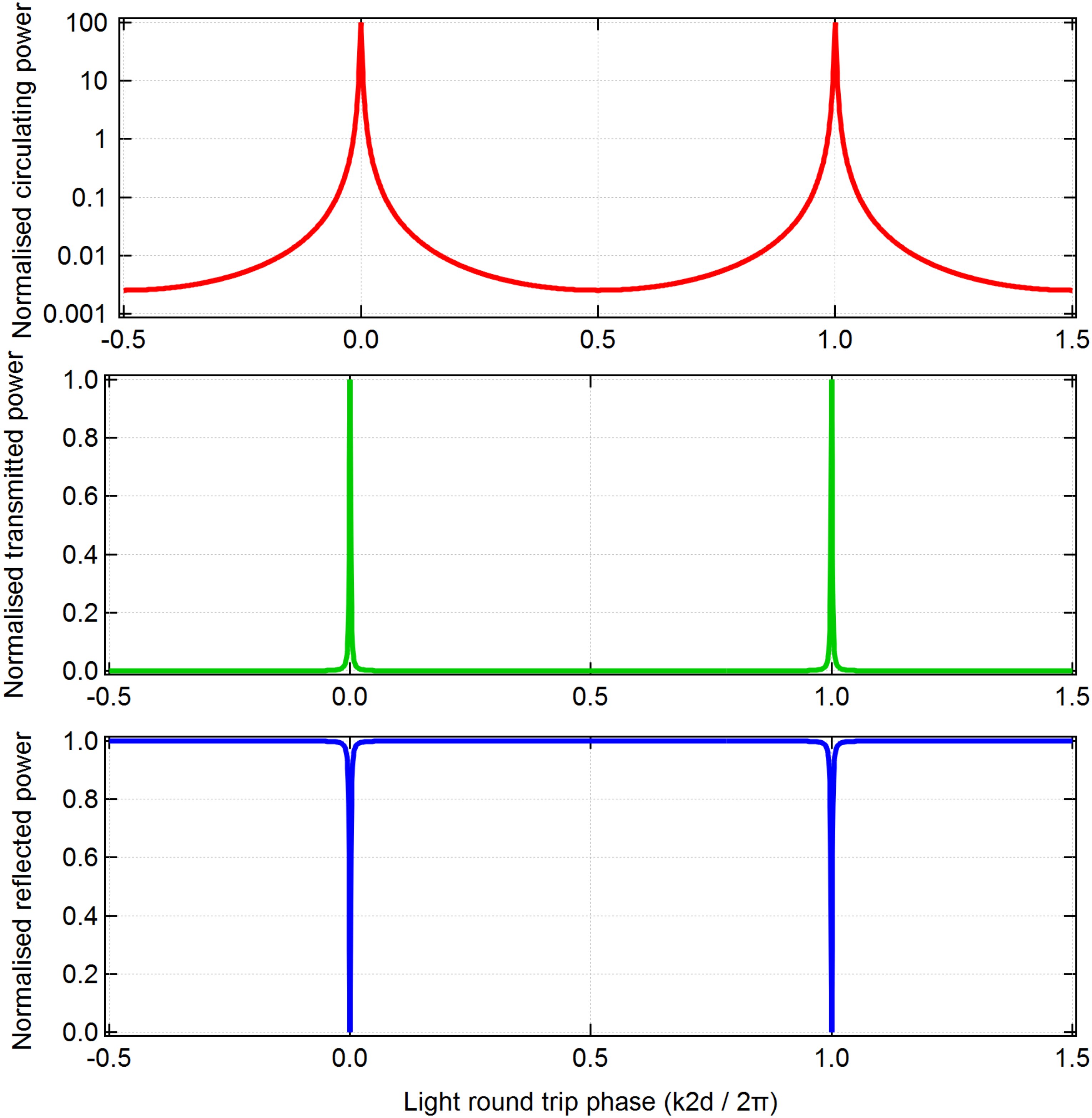
$$E_{circ} = \frac{i t_{IM}}{1 - r_{IM} r_{EM} e^{-ik2d}} E_0$$

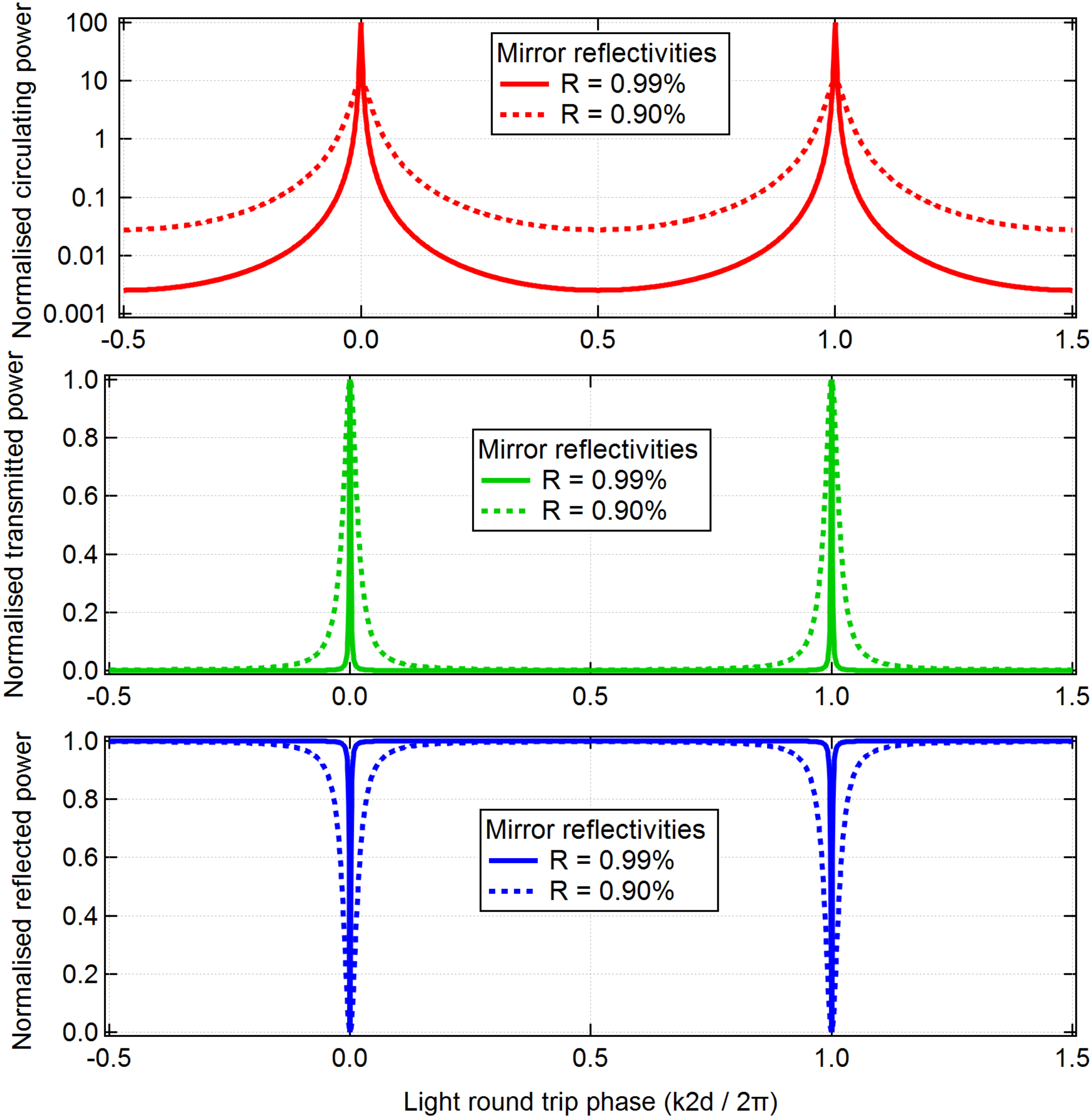
$$E_{trans} = \frac{-t_{IM} t_{EM} e^{-ik2d}}{1 - r_{IM} r_{EM} e^{-ik2d}} E_0$$

$$E_{ref} = \left(r_{IM} - \frac{t_{IM}^2 r_{EM} e^{-ik2d}}{1 - r_{IM} r_{EM} e^{-ik2d}} \right) E_0$$

Circulating power a function of the detuning




Normalised circulating power
100
10
1
0.1
0.01
0.001
-0.5
0.0
0.5
1.0
1.5
Normalised transmitted power
1.0
0.8
0.6
0.4
0.2
0.0
Normalised reflected power
Light round trip phase (k2d / 2π)



Normalised circulating power
100
10
1
0.1
0.01
0.001
Mirror reflectivities
R = 0.99%
R = 0.90%
-0.5
0.0
0.5
1.0
1.5
Normalised transmitted power
1.0
0.8
0.6
0.4
0.2
0.0
Mirror reflectivities
R = 0.99%
R = 0.90%
Normalised reflected power
Mirror reflectivities
R = 0.99%
R = 0.90%
Light round trip phase (k2d / 2π)

Some key figures

The cavity gain $$G = \frac{T_{IM}}{\left(1 - \sqrt{R_{IM} R_{EM}}\right)^2}$$

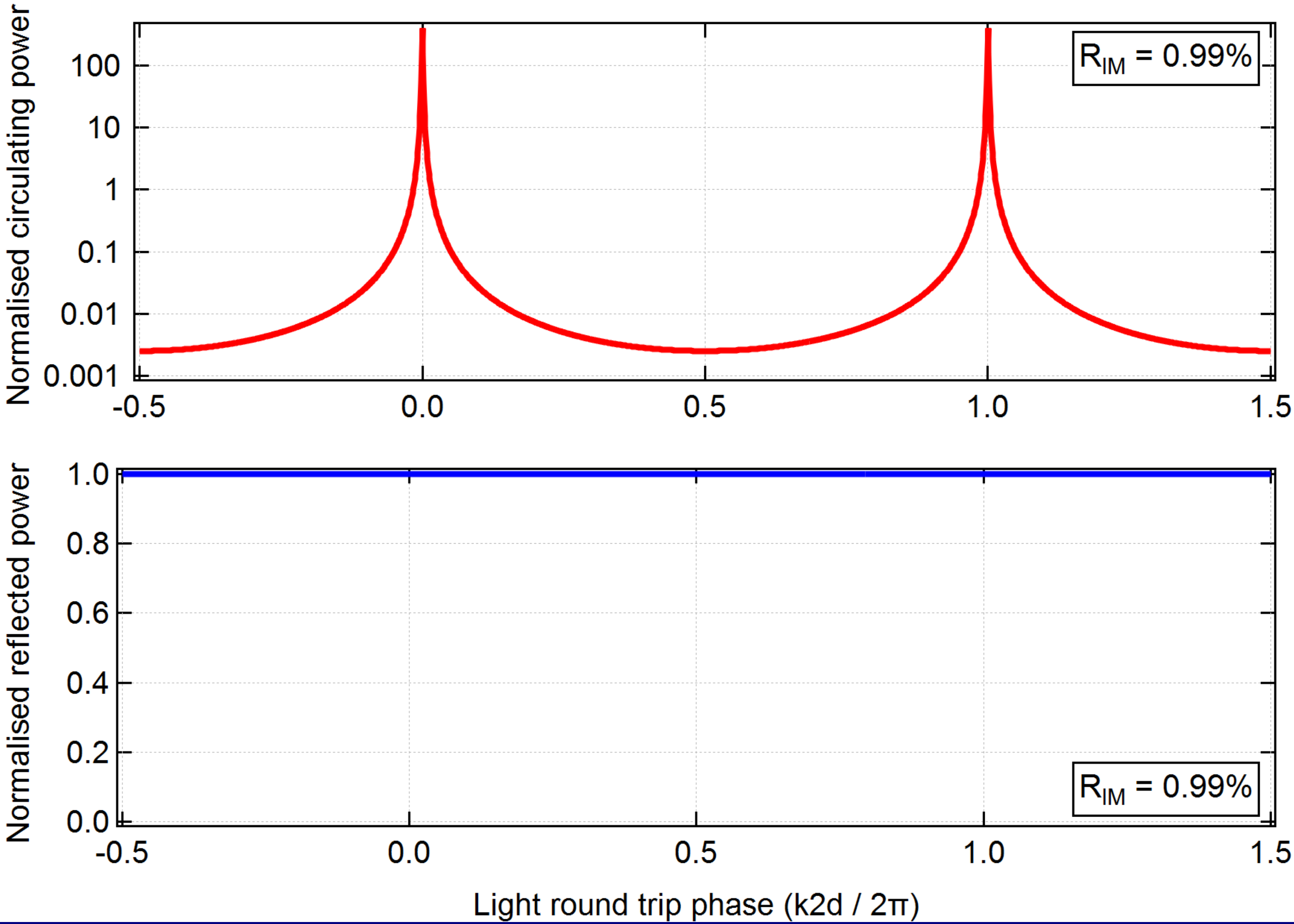
The finesse $$\mathfrak{F} = \frac{\pi \sqrt[4]{R_{IM} R_{EM}}}{1 - \sqrt{R_{IM} R_{EM}}}$$

The FSR $$\frac{c}{2L}$$

The FWHM $$\frac{FSR}{\mathfrak{F}}$$

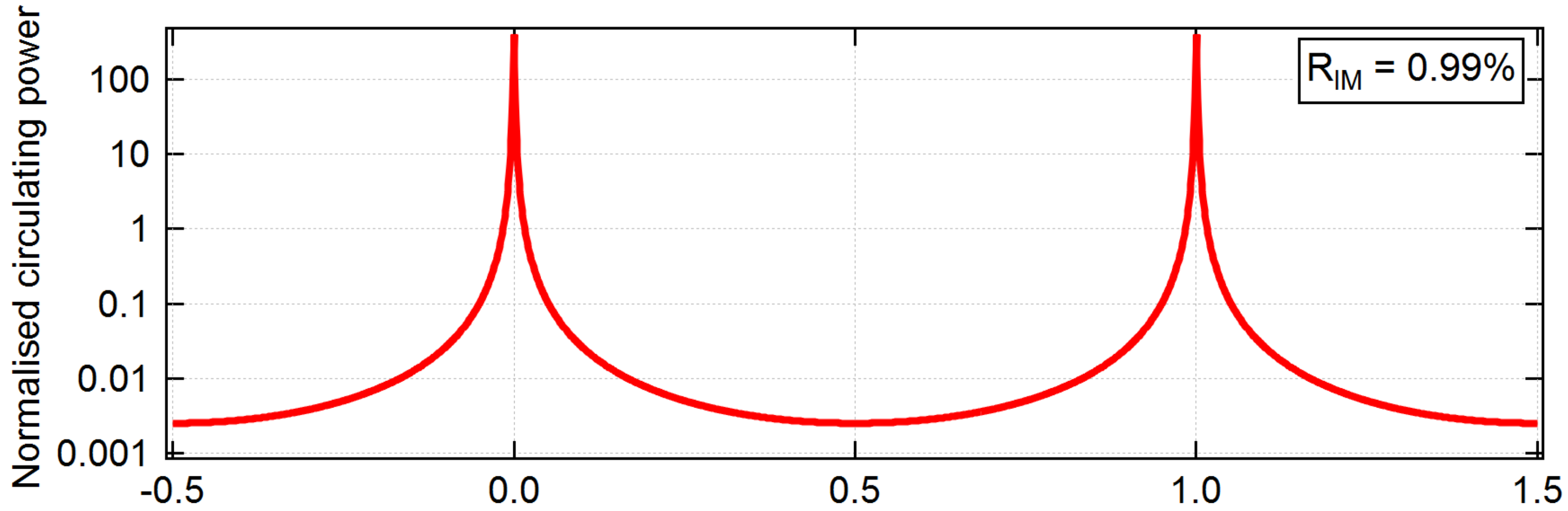
A particular case, a very reflective end mirror

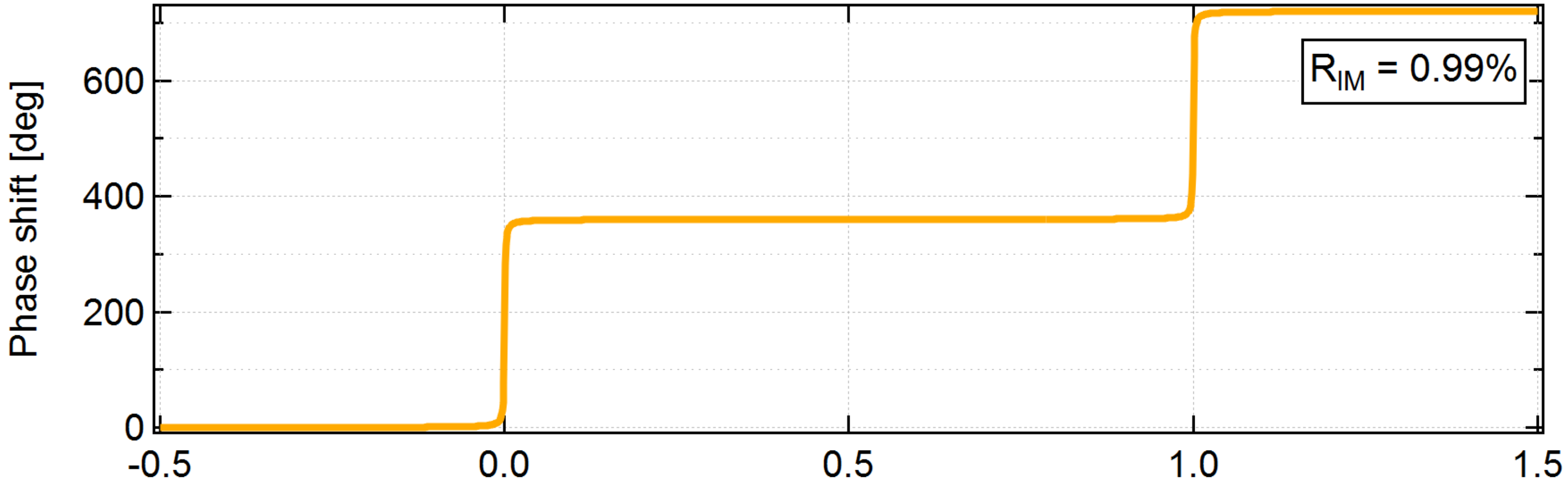
This is the case for GW detector (T_{EM} = 4-5 ppm)



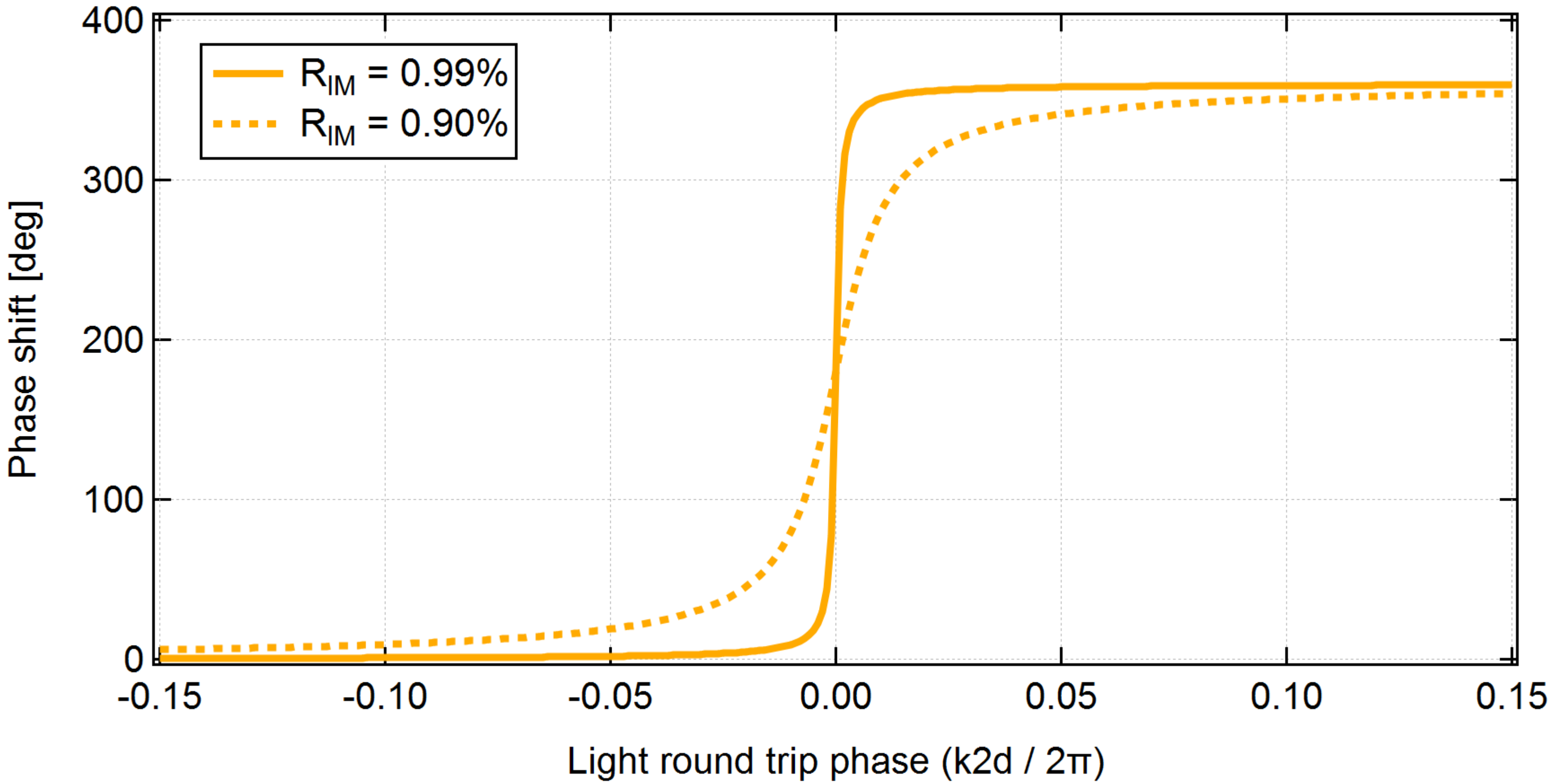
Yes, and why does that help ?

It is all in the phase in reflection !





Phase shift for different finesse



$$G = \frac{4}{T_{IM}}$$

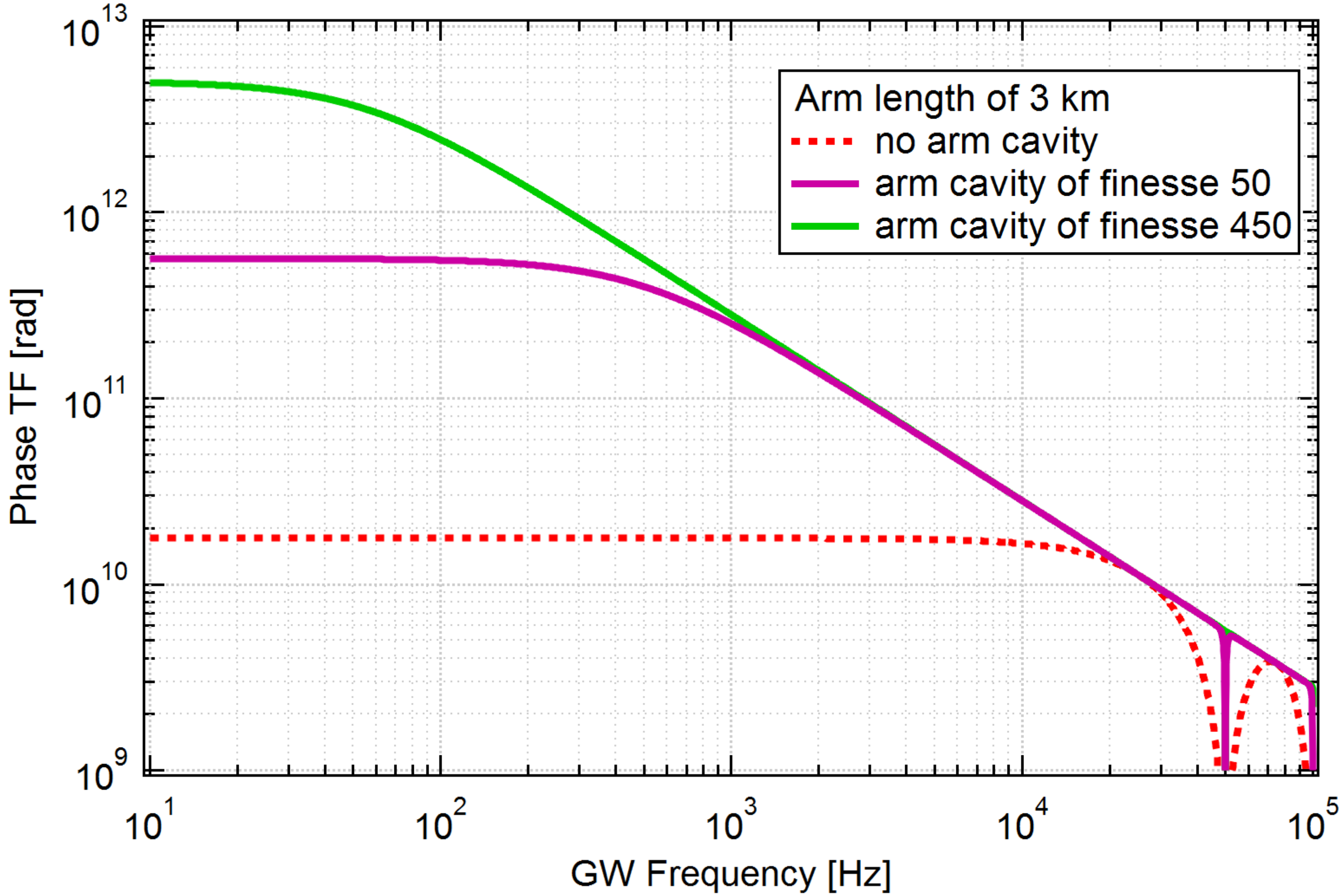
$$\mathfrak{F} = \frac{2\pi}{T_{IM}}$$

Amplification of the phase shift by a factor : $\dfrac{2\mathfrak{F}}{\pi}$

Typical Fabry-Perot cavity finesses

Type of experiments	Finesse
Laser analysers	~100
Reference cavities / atomic clock	~100 000
First generation GW detector	50
Second generation GW detector	450
Thrid generation GW detector	~900

The phase transfer function with FP cavity



- Increase the sensitivity
- But further reduction of the bandwidth $f_c = \dfrac{c}{4\mathfrak{F}L}$

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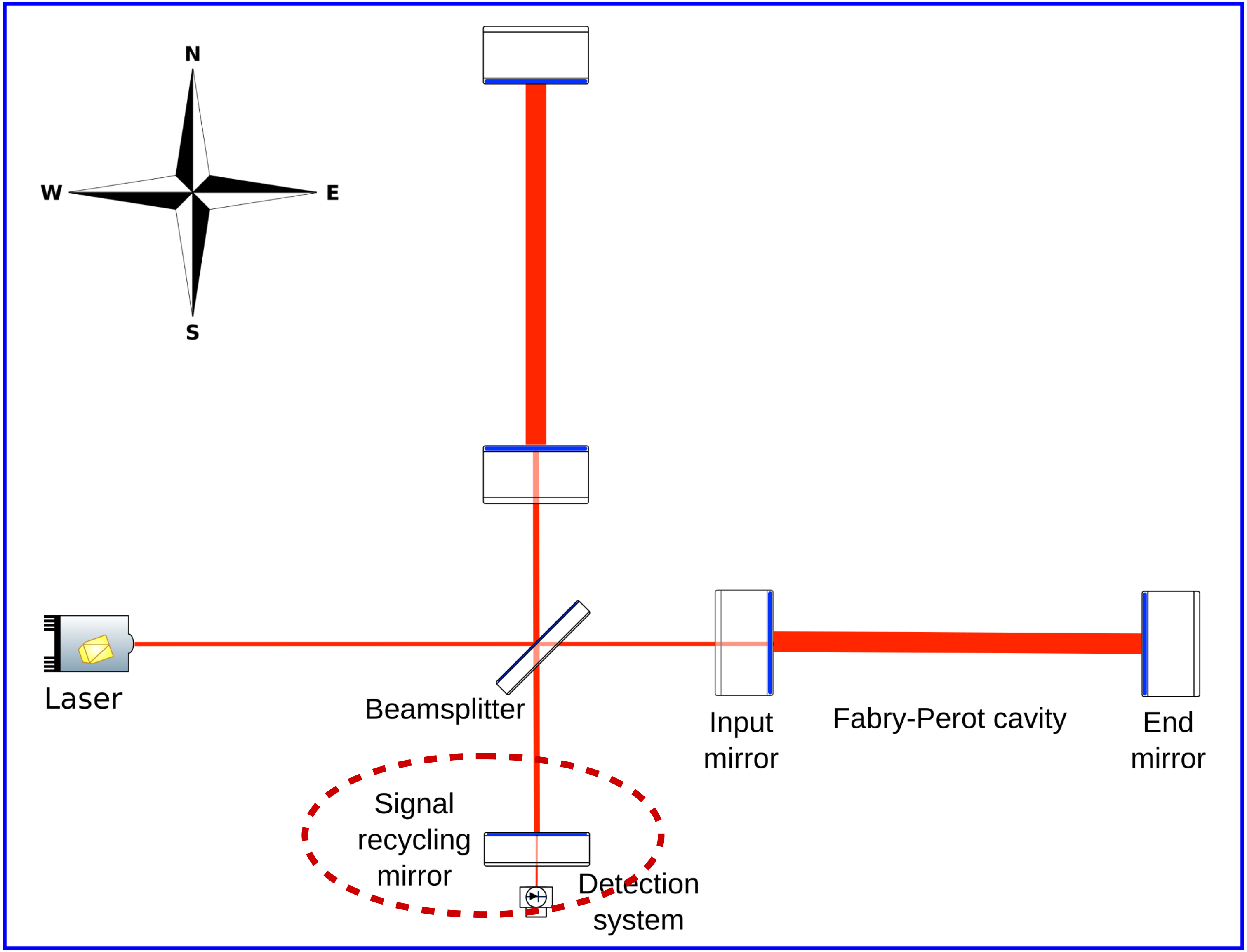
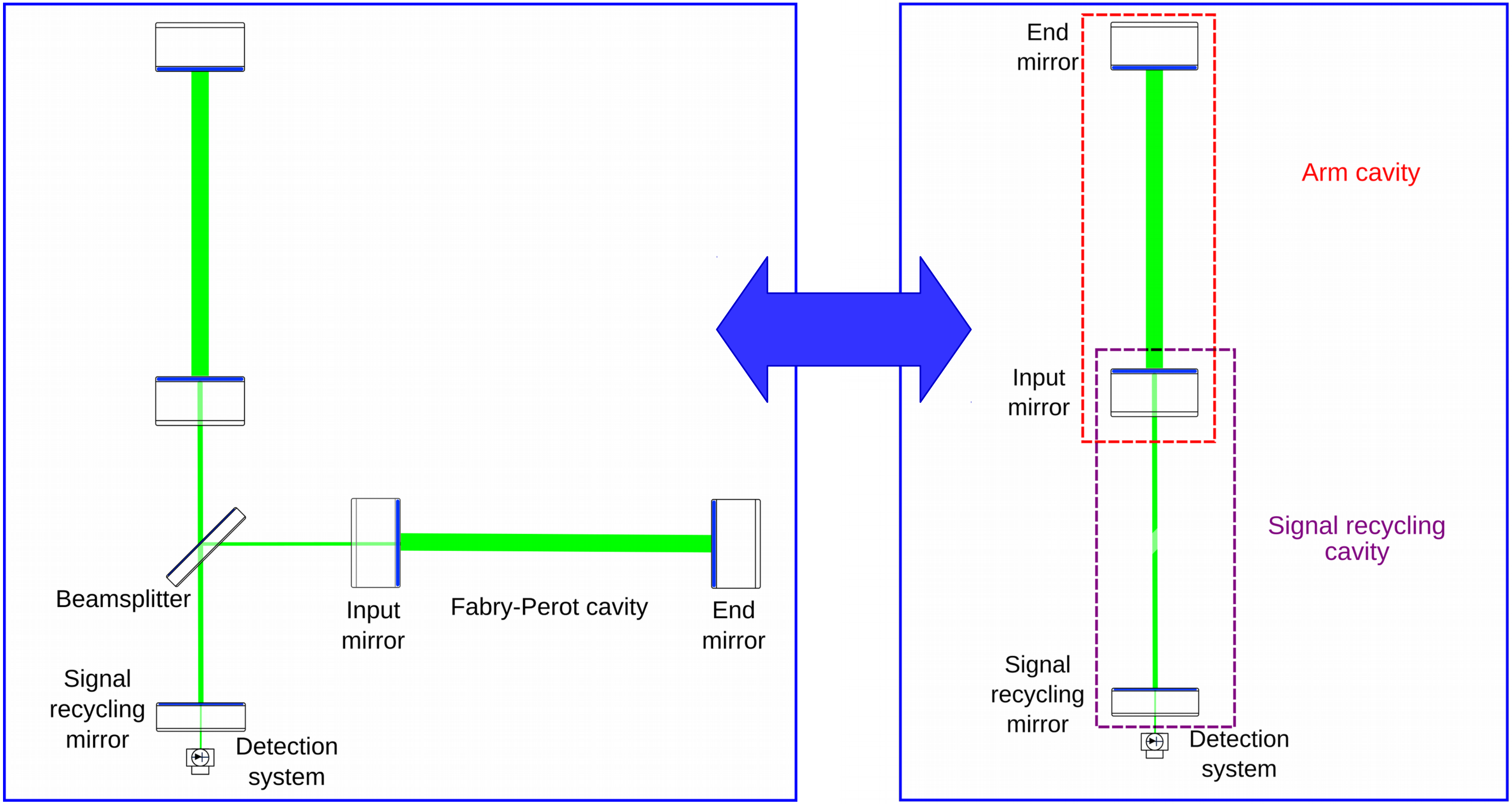
The signal recycling cavity



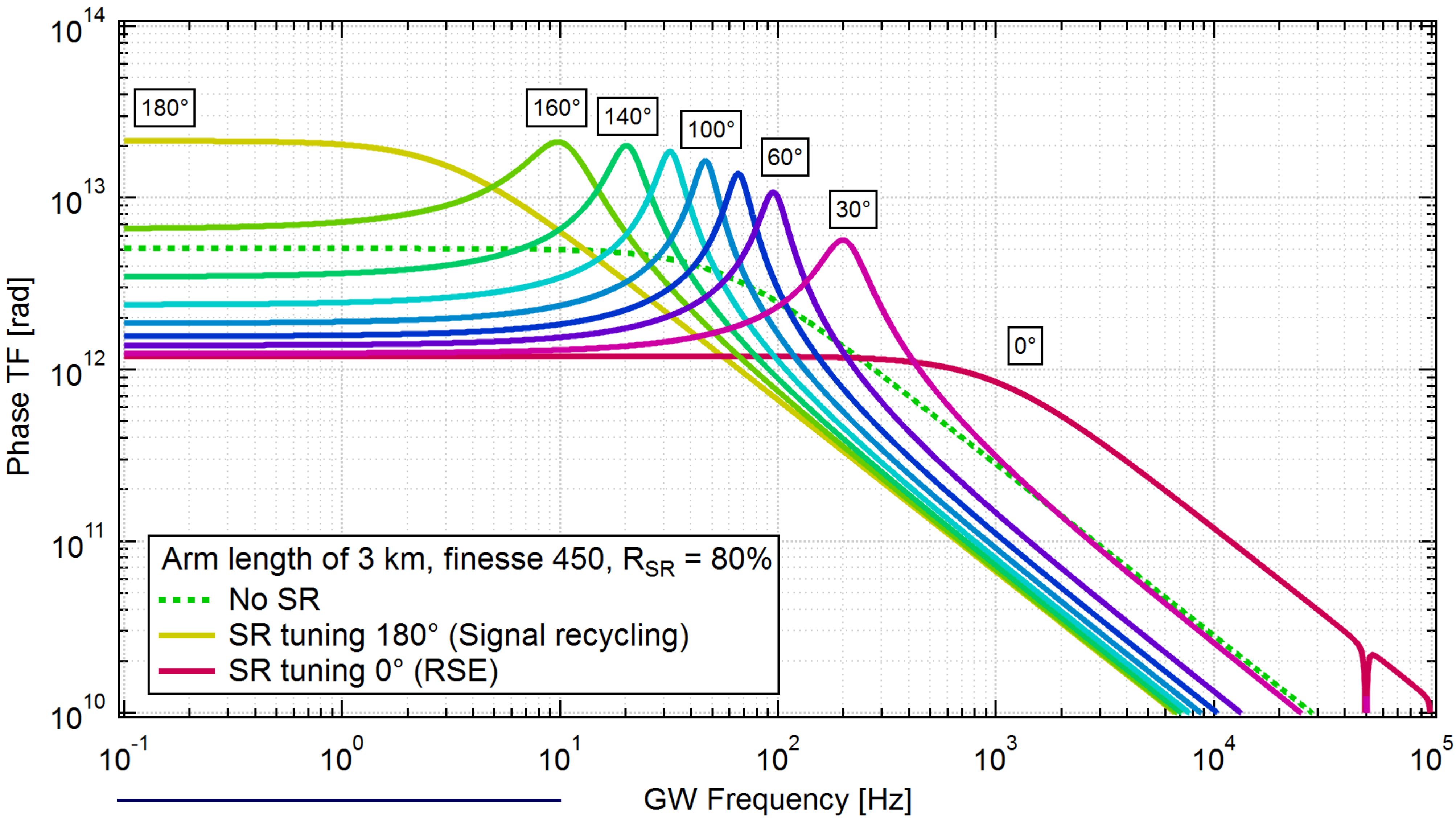
New mirror at the output port

For the GW signal sidebands



Possibility to tune the apparent transmission of input mirror for the GW signal

Tuning the response of the interferometer



(very low frequencies)

III

The sensitivity of the Michelson

1. Fundamental quantum noise
2. Power recycled Michelson

How to quantify the noise ?

Use the power spectral density : S_V

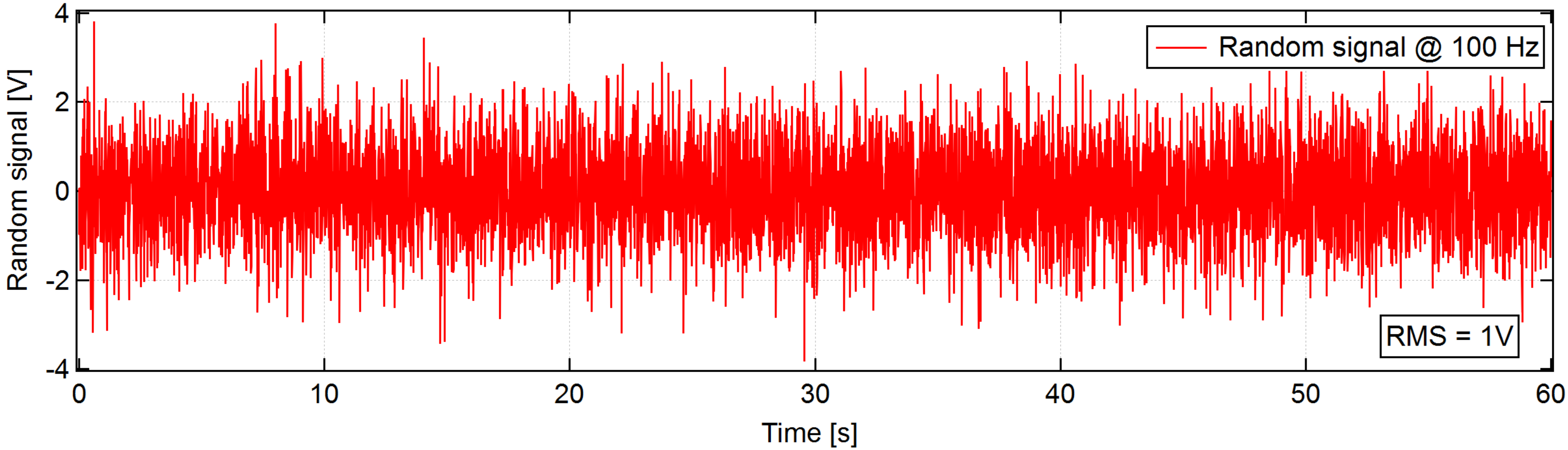
$$S_V(\omega) = \lim_{T \to \infty} \frac{1}{2T} \left| \int_{-T}^{+T} V(t) e^{-i\omega t} dt \right|^2$$

In unit of : $\frac{[V]^2}{Hz}$, that represents the noise power density in a given bandwidth as a function of the frequency.

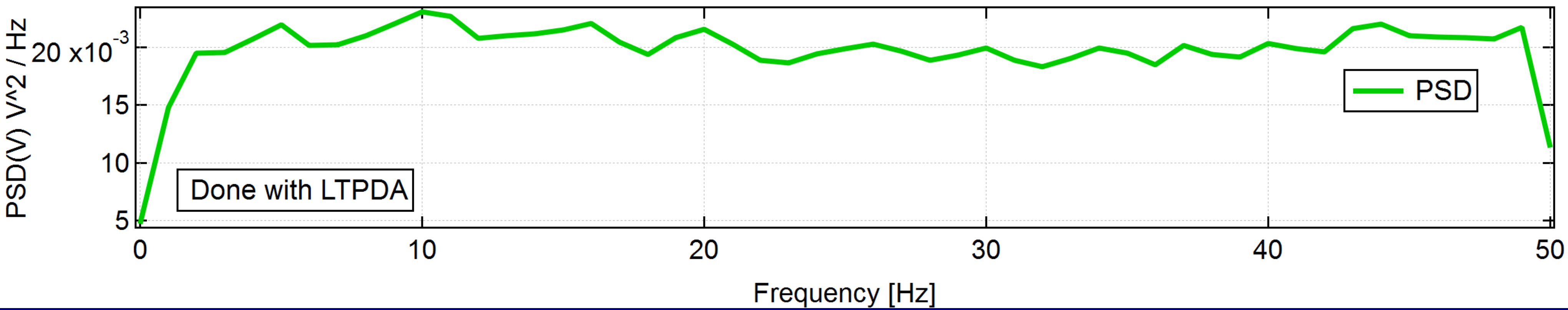
We use also the noise amplitude spectral density : $\sqrt{S_V(\omega)}$

is also the Fourrier transform of the auto correlation function

Ok, that definition does not really help!

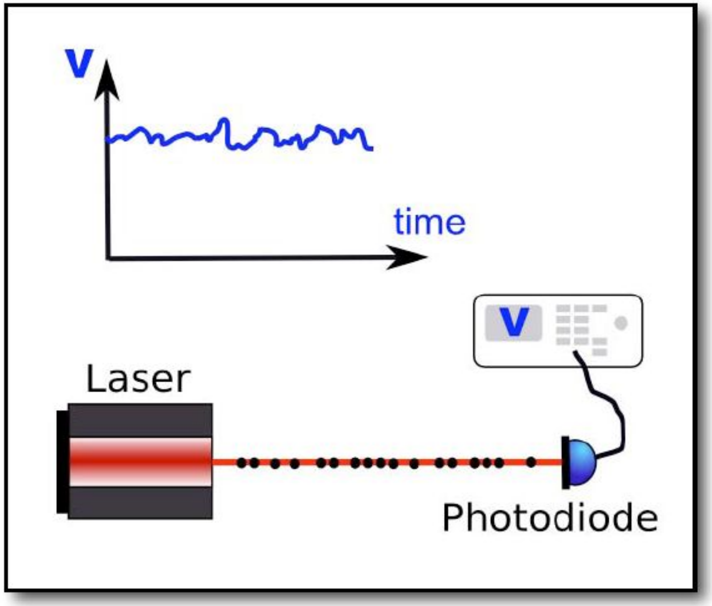


$$\mathrm{RMS_V} = \lim_{T\to\infty} \sqrt{\frac{1}{T}\int_0^T V^2(t)\,dt} \qquad \mathrm{RMS_V^2} = \int_0^\infty S_V(\omega)\,d\omega$$



The intrinsic shot noise

Measuring an optical power is counting the number of photon for a given time.



Due to the discrete nature of light, arrival time of photons follows a Poisson statistics

$$S_{SN}(\omega) = 2Ph_p\frac{c}{\lambda}$$

Image credit: Stefan Hild, IGR, Glasgow

Shot noise equivalent to h

Optical power at the output of the interferometer :

$$P^{DP} = \frac{P_0}{2}\left(1 + \cos(4kL_-)\right)$$

In presence of a GW signal :

$$P^{DP} = \frac{P_0}{2}\left(1 + \cos(4k(L_- + \frac{L_+}{2}h_0(t))\right)$$

2 contributions, a DC term and a (small) signal term :

$$S_{SN}^{DP}(\omega) = P_0 h_p \frac{c}{\lambda}(1 + \cos(\beta))$$

$$S_h^{DP}(\omega) = P_0^2 k^2 L^2 \sin^2(\beta) S_h(\omega)$$

With : $\beta = 4kL_-$

Shot noise equivalent to h

The minimum signal we can detect is when the PSD of the shot noise is equal to the PSD of the signal :

$$\frac{S_h^{DP}(\omega)}{S_{SN}^{DP}(\omega)} = 1$$

Which gives:

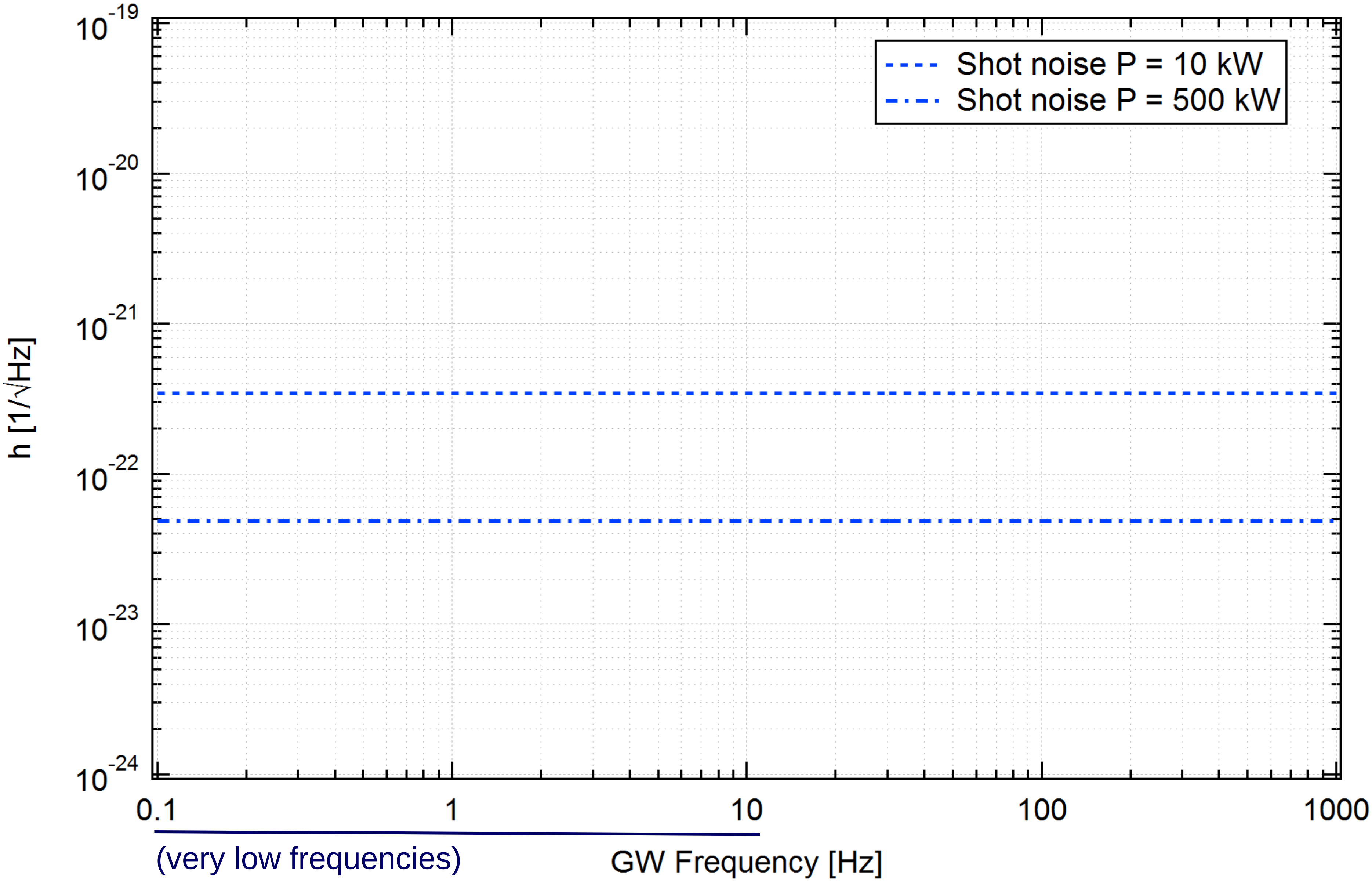
$$S_h^{min}(\omega) = \frac{1}{(2\pi L_+)^2} \frac{h_p \lambda c}{P_0}$$

Very good aproximation but not an exact one

(missing a factor 2)

$$\sqrt{S_h^{min}(\omega)} = \frac{2 \times 10^{-20}}{\sqrt{P_0}} [1/\sqrt{Hz}]$$

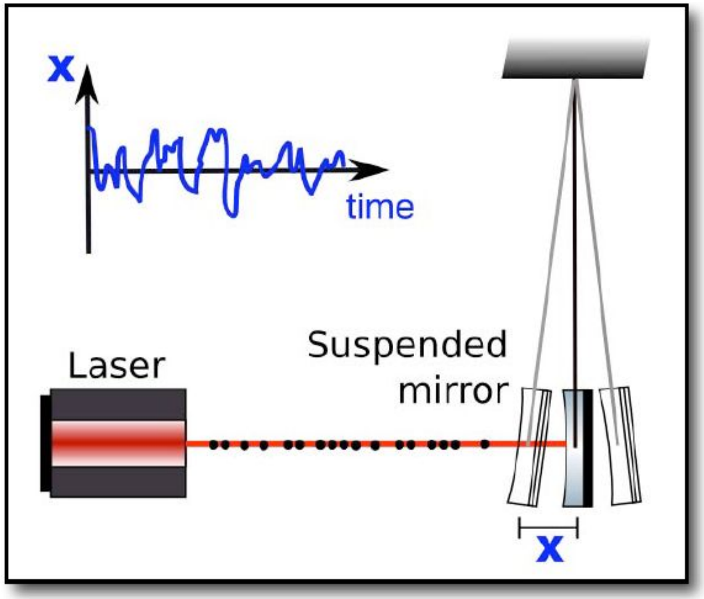
Shot noise limited (simple) Michelson



(very low frequencies)

Radiation pressure noise

Measuring the mirror position with light, induced a back action : the radiation pressure noise.

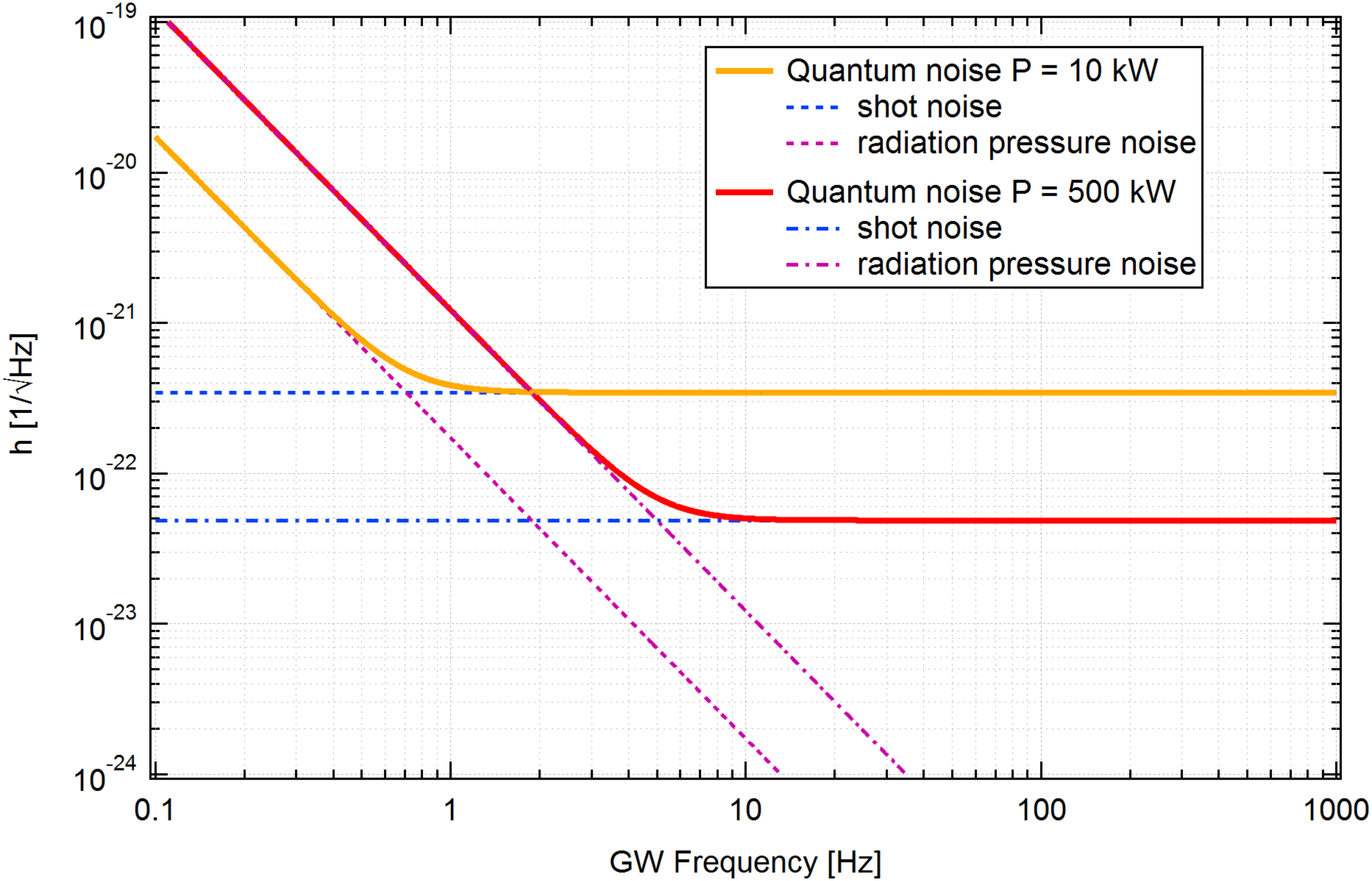


$$F = 2\frac{P_{inc}}{c}$$

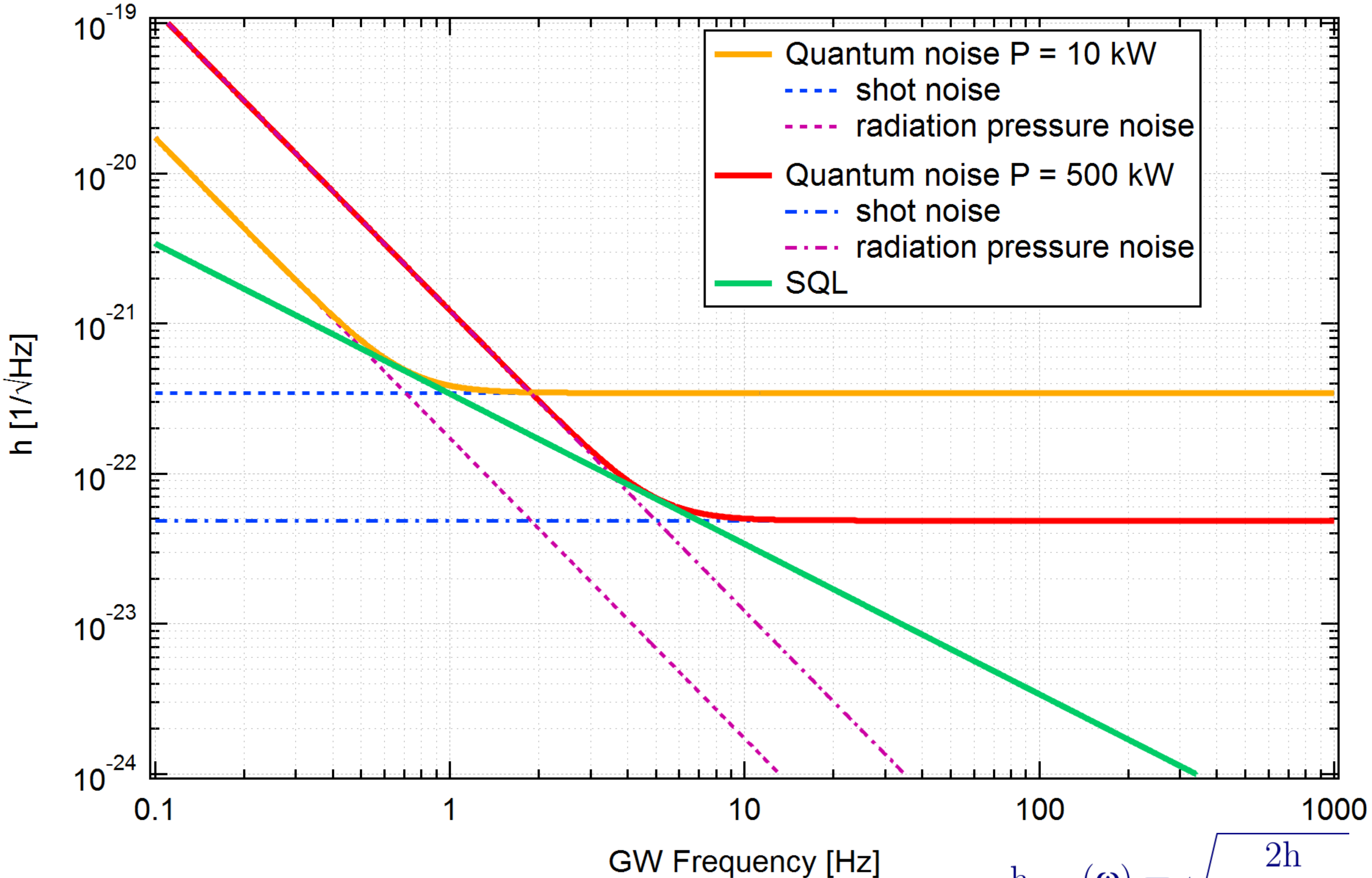
PSD for a simple Michelson :

$$S_{RP}(\omega) = \frac{1}{mL\omega^2}\sqrt{\frac{4hP}{c\lambda}}$$

Shot noise limited (simple) Michelson

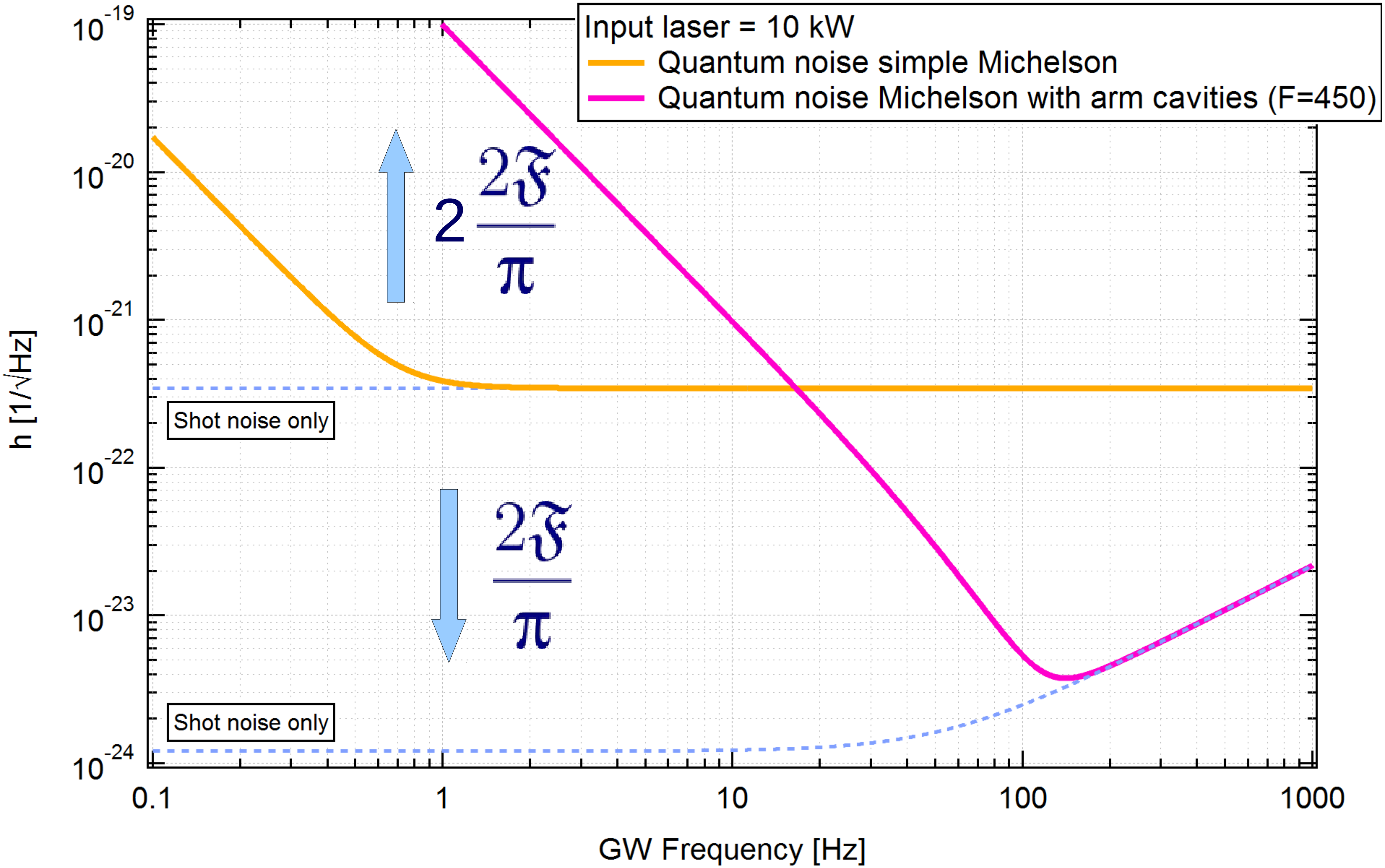


Shot noise limited (simple) Michelson



$$h_{SQL}(\omega) = \sqrt{\frac{2\hbar}{\pi m L^2 \omega^2}}$$

Quantum noise with FP arm cavities

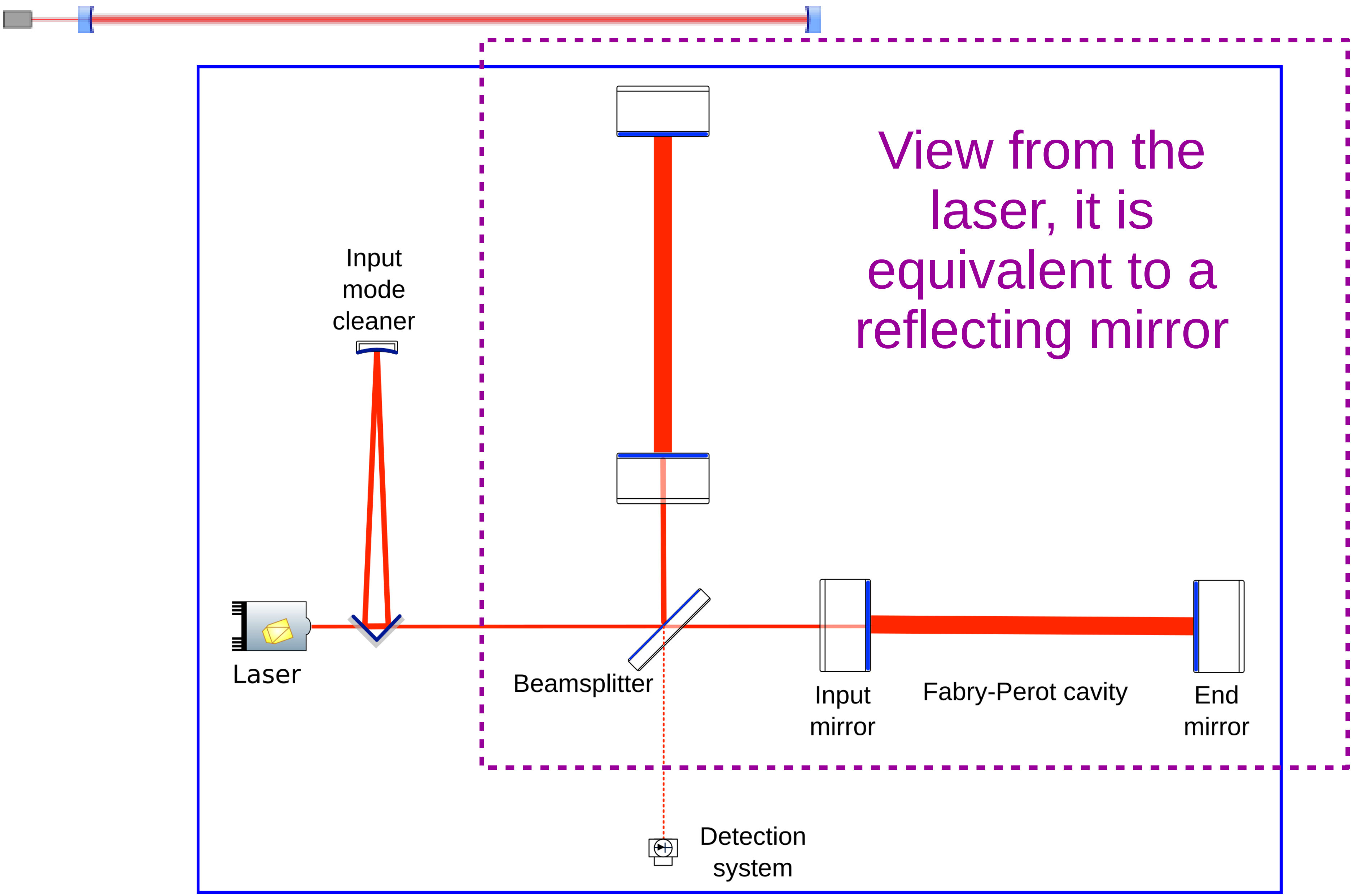


III

The sensitivity of the Michelson

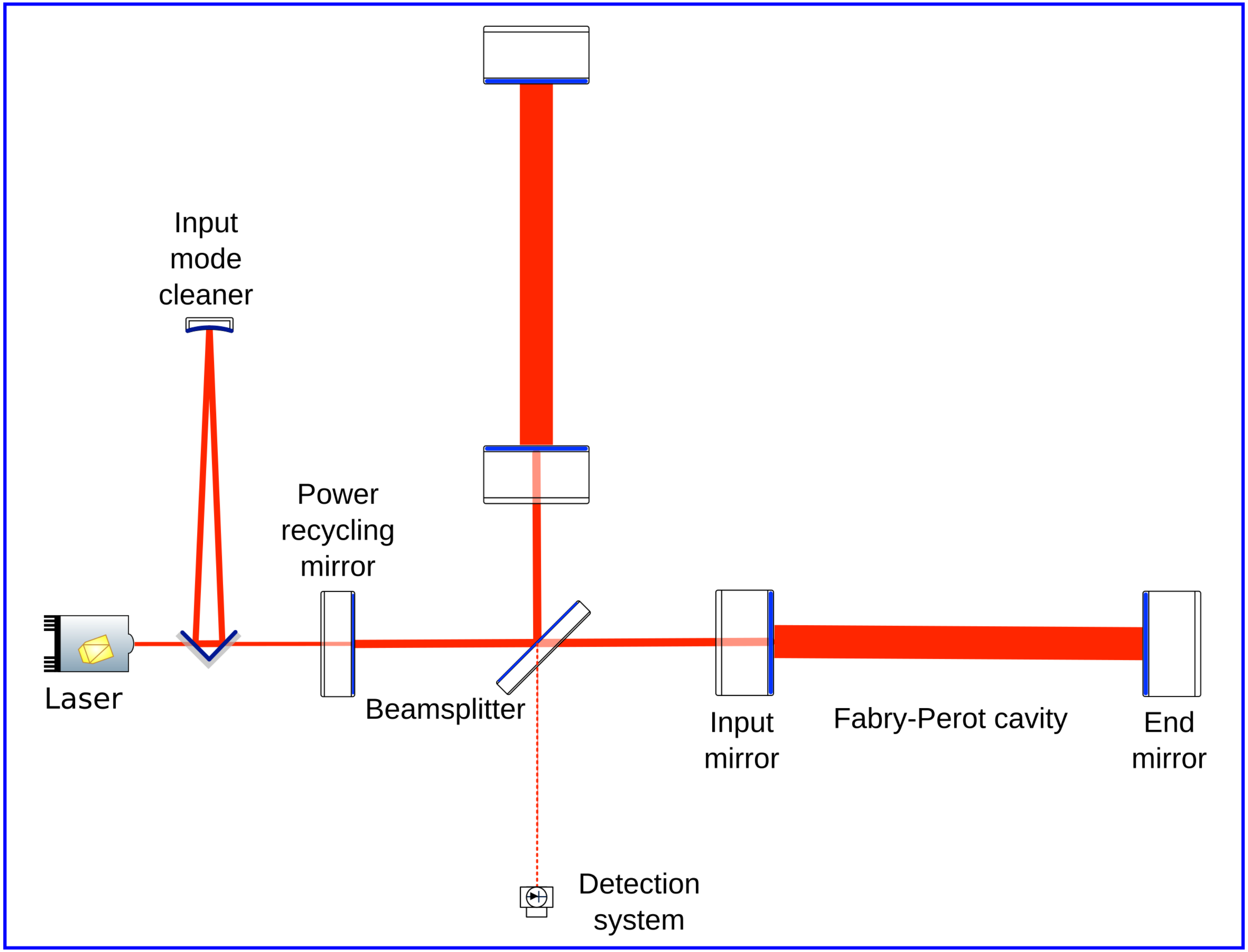
1. Fundamental quantum noise
2. Power recycled Michelson

Do not waste any light!



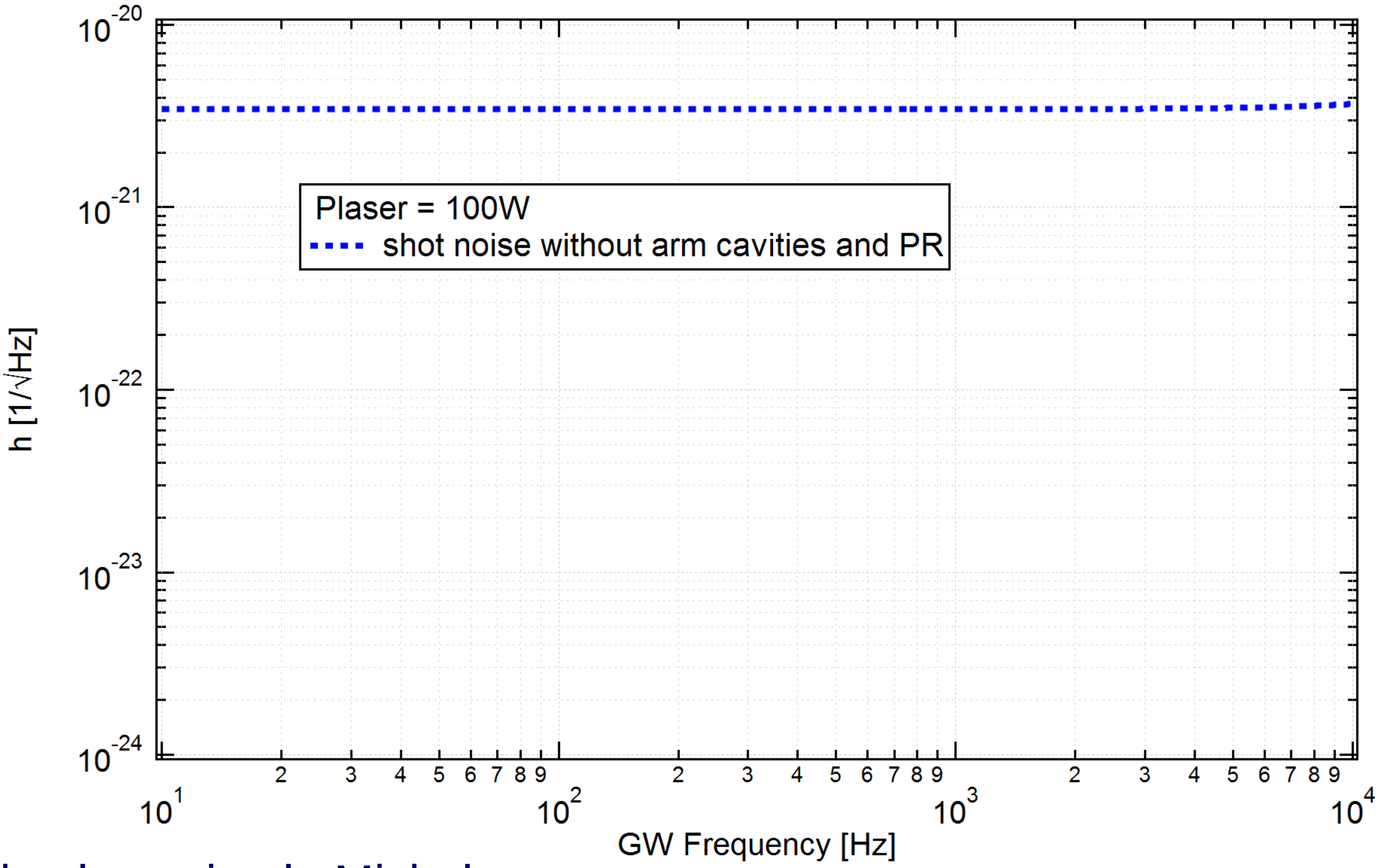
View from the laser, it is equivalent to a reflecting mirror

Adding a new mirror



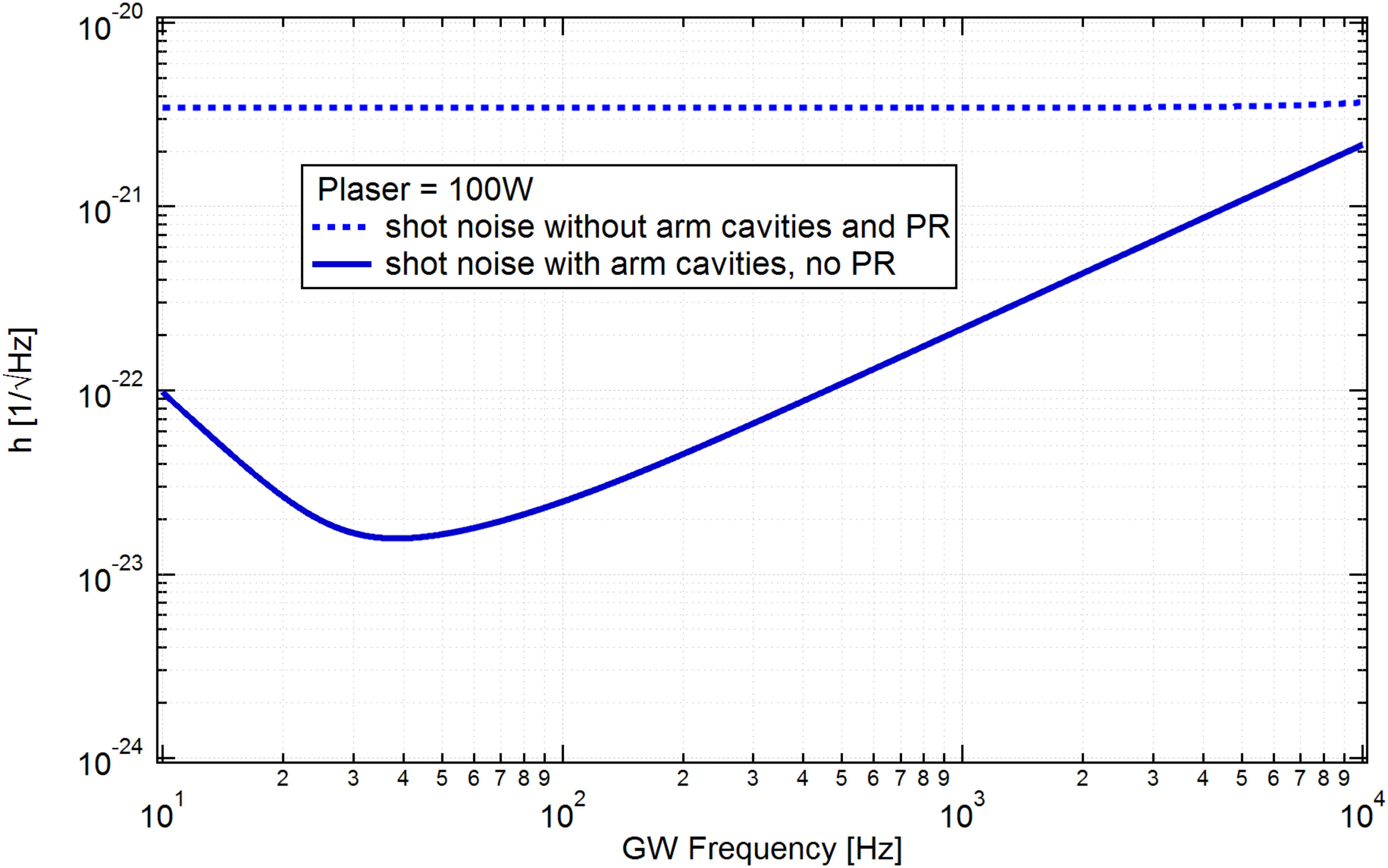
In real interferometer, power gain ~ 40

Updated sensitivity



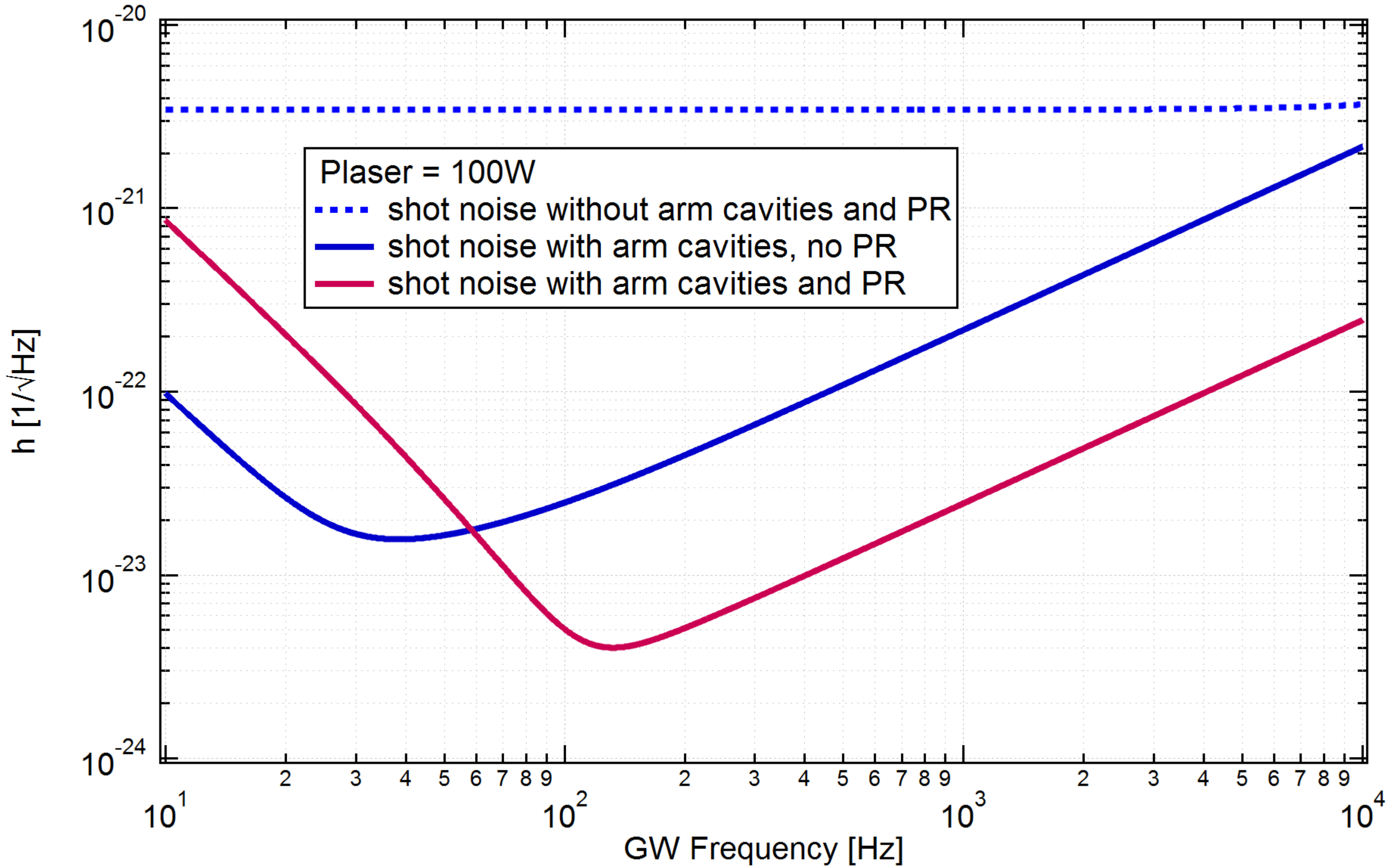
3 km long simple Michelson

Updated sensitivity



3 km long simple Michelson with arm cavities (F=450)

Updated sensitivity



3 km long simple Michelson with arm cavities (F=450) and PR

IV

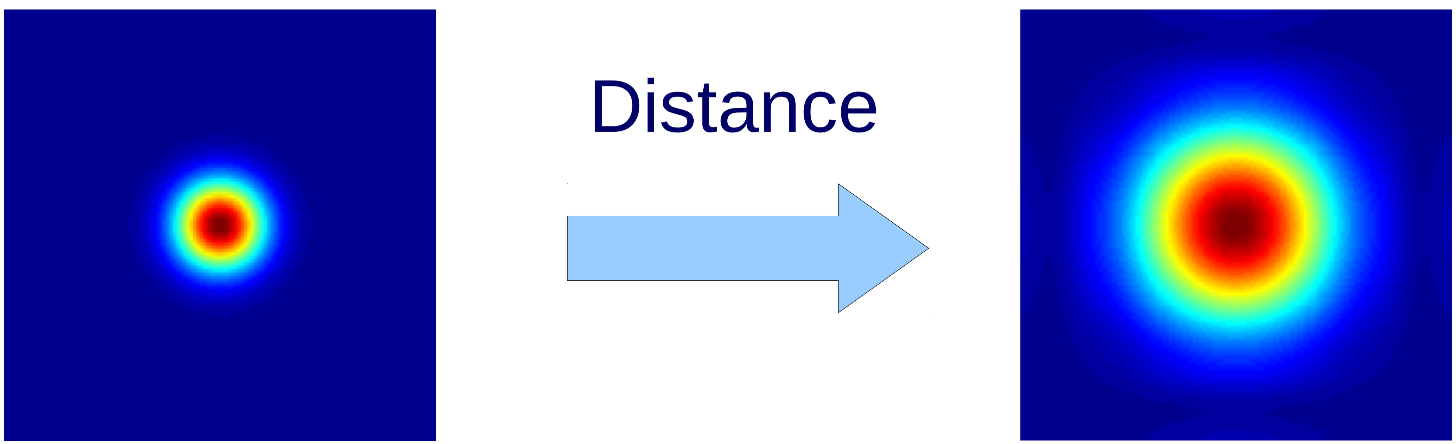
Gaussian beams and Fabry-Perot cavities

(if time permits)

Gaussian beam

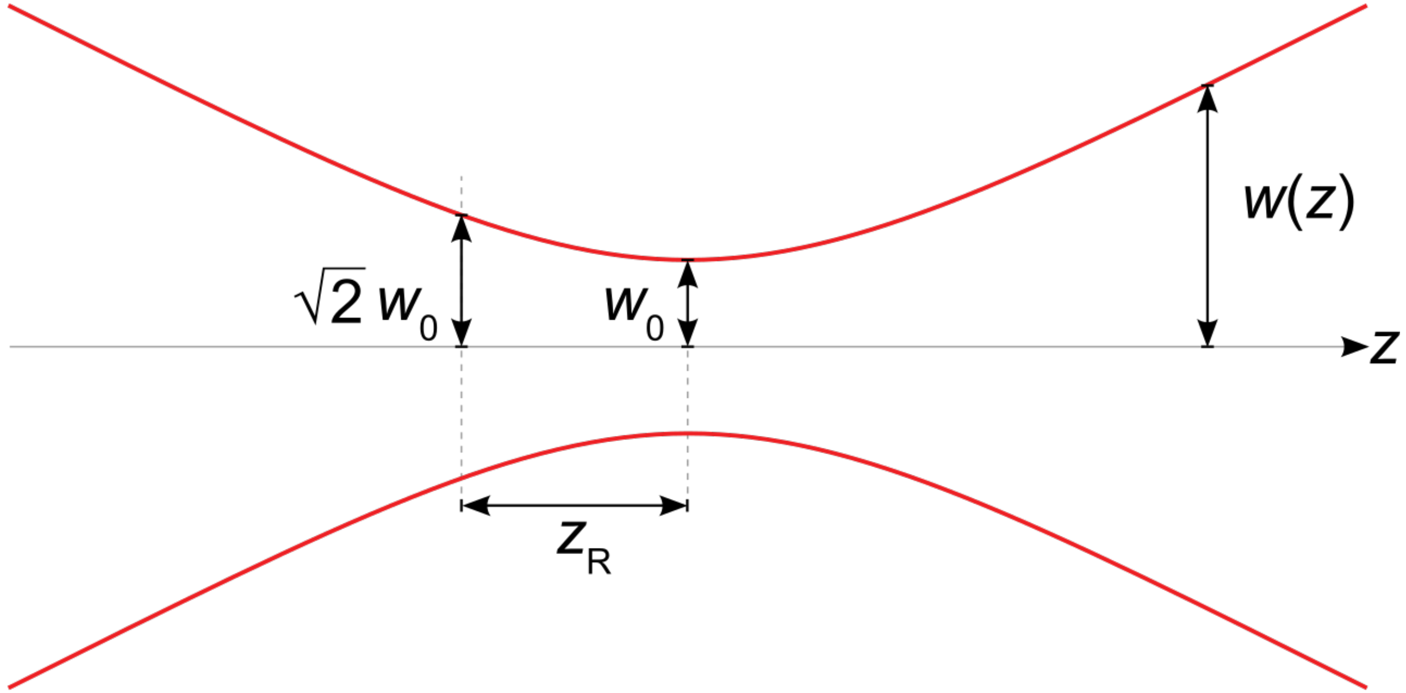
The laser beam circulating in the interferometer is a gaussian beam

(eigen mode of the free space propagation)



Gaussian beam

Evolution of the beam radius :



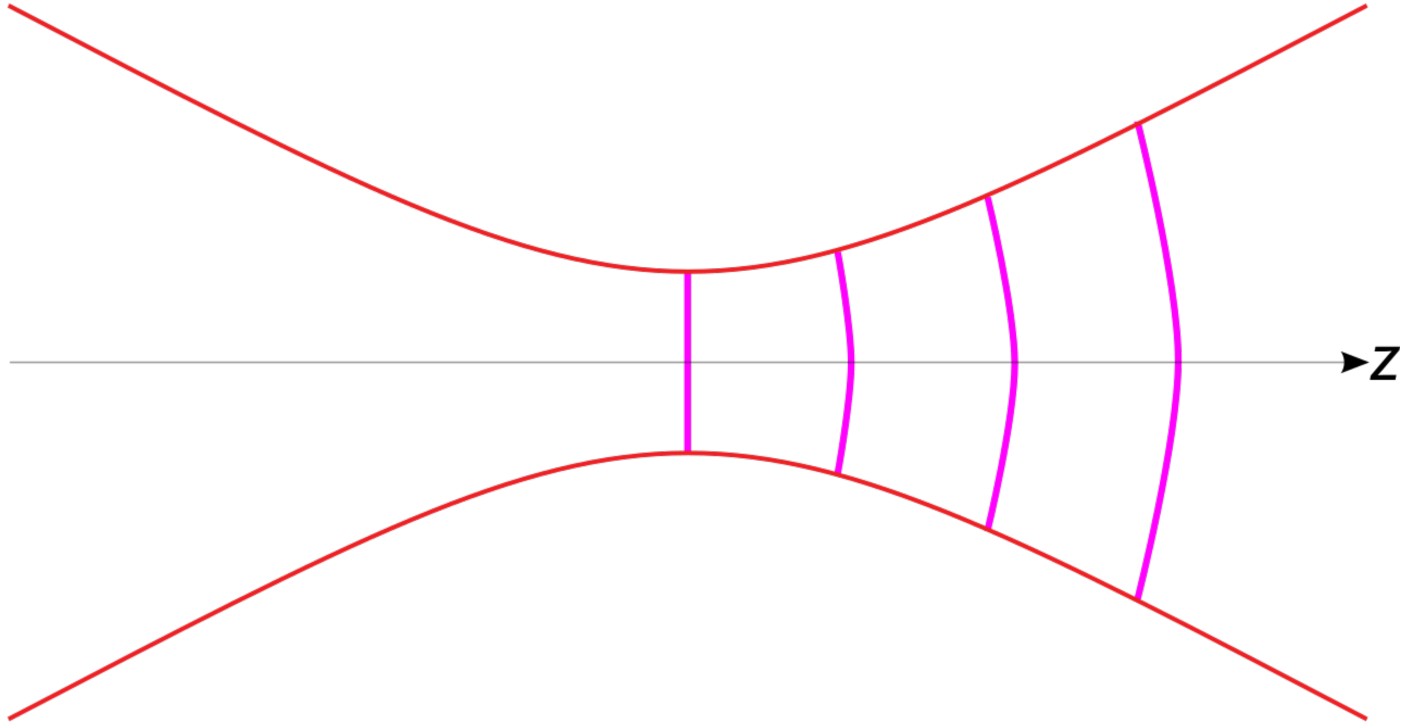
$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2} \qquad z_R = \frac{\pi w_0^2}{\lambda}$$

A minima in the beam size at the waist

Image taken (and then modified): from Wikipedia

Gaussian beam

A second paramater : the complex wavefront curvature



$$R(z) = z\left(1 + \left(\frac{z_R}{z}\right)^2\right) \qquad z_R = \frac{\pi w_0^2}{\lambda}$$

Infinite at the waist (flat wavefront)

Image taken (and then modified): from Wikipedia

The more complex picture

Gouy phase shift

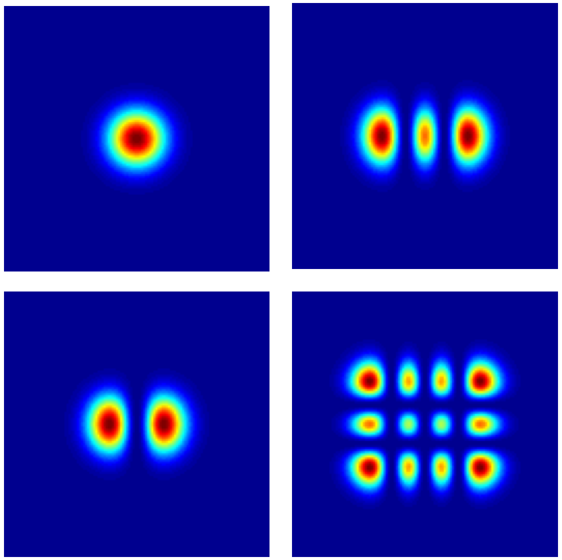
Mathematical description of the beam :

$$E(r,z) = E_0(r,z)e^{\left(-\frac{r^2}{w^2(z)}\right)}e^{\left(-i\left(kz+k\frac{r^2}{2R(z)}-\psi(z)\right)\right)}$$

Amplitude envelope

Phase shit from propagation

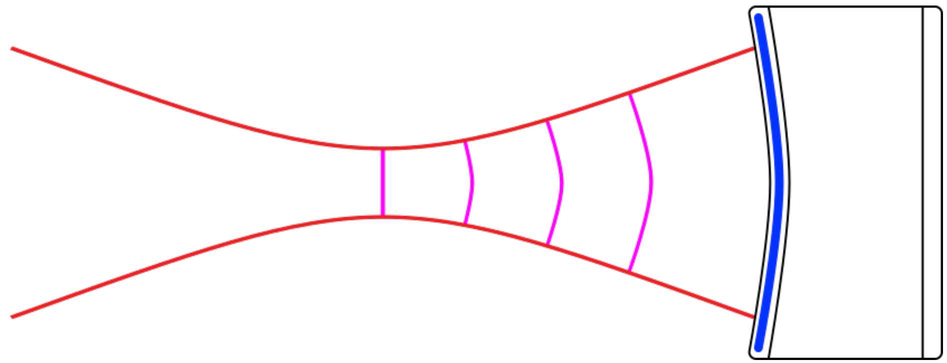
Phase shit from wavefront



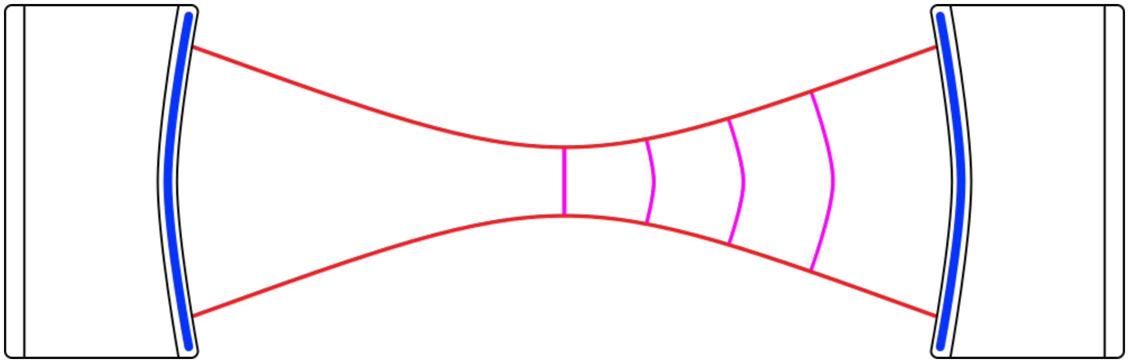
Not only the fundamental mode !

FP Cavities and Gaussian beam

Adding a mirror to reflect the beam on its path:

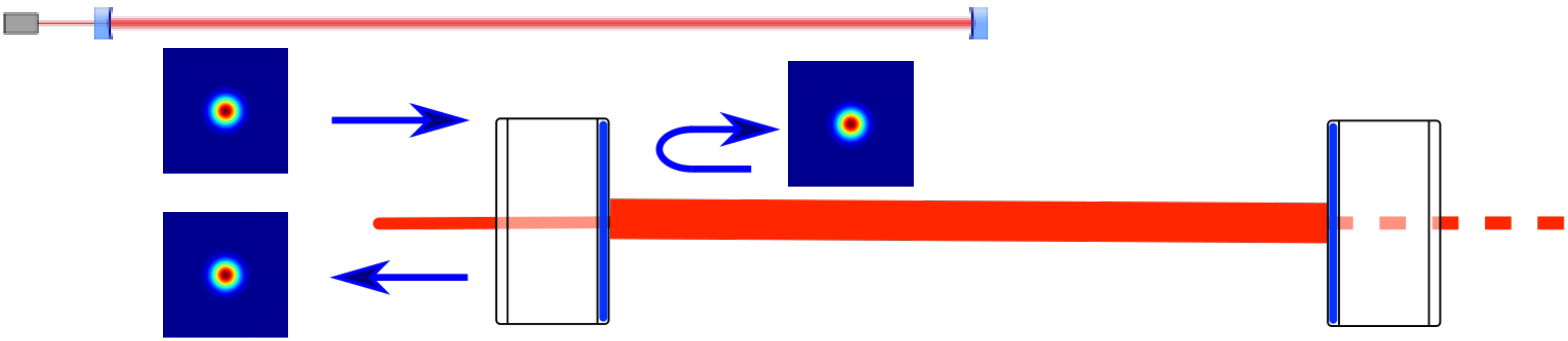


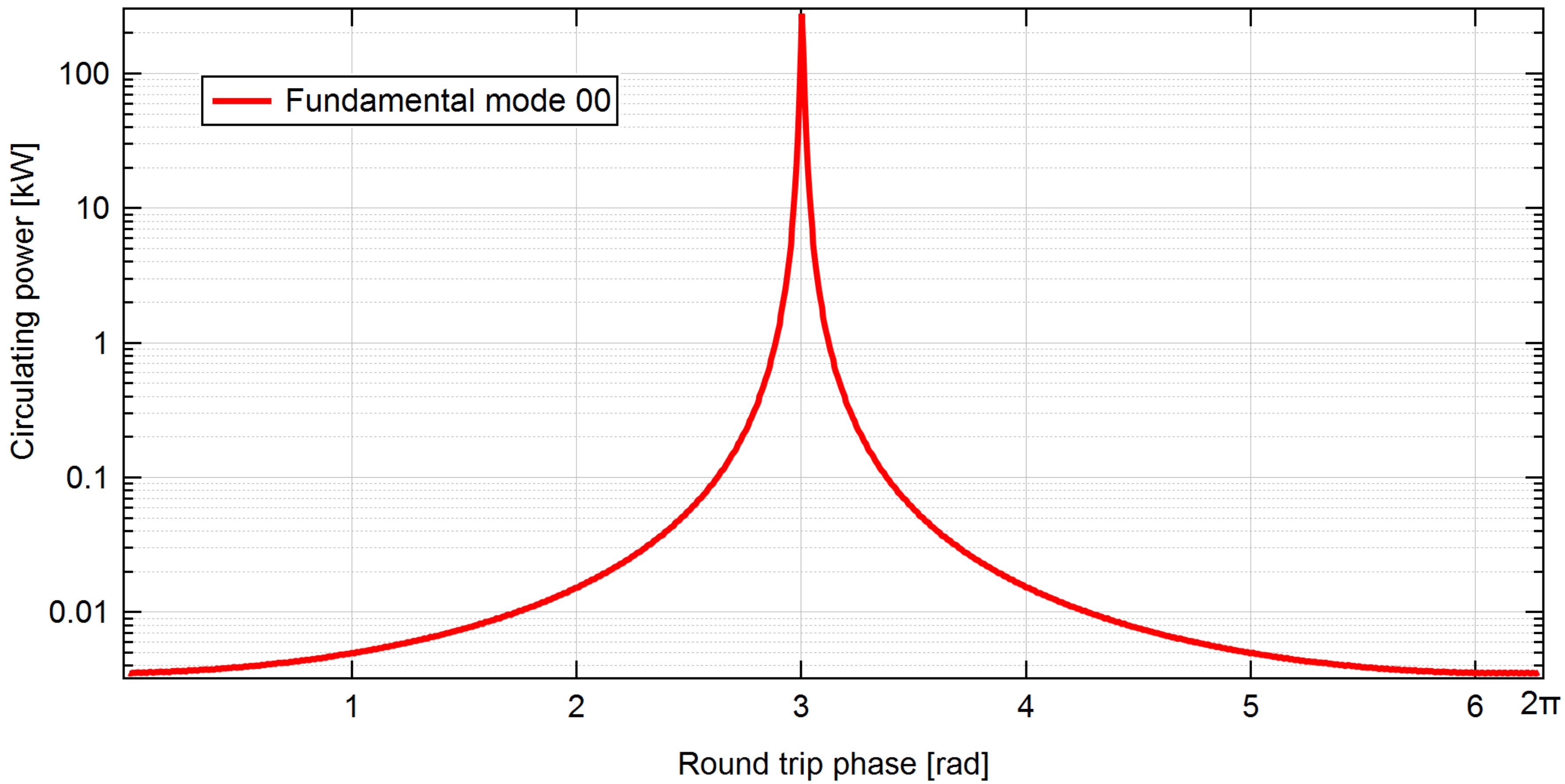
And a second one:



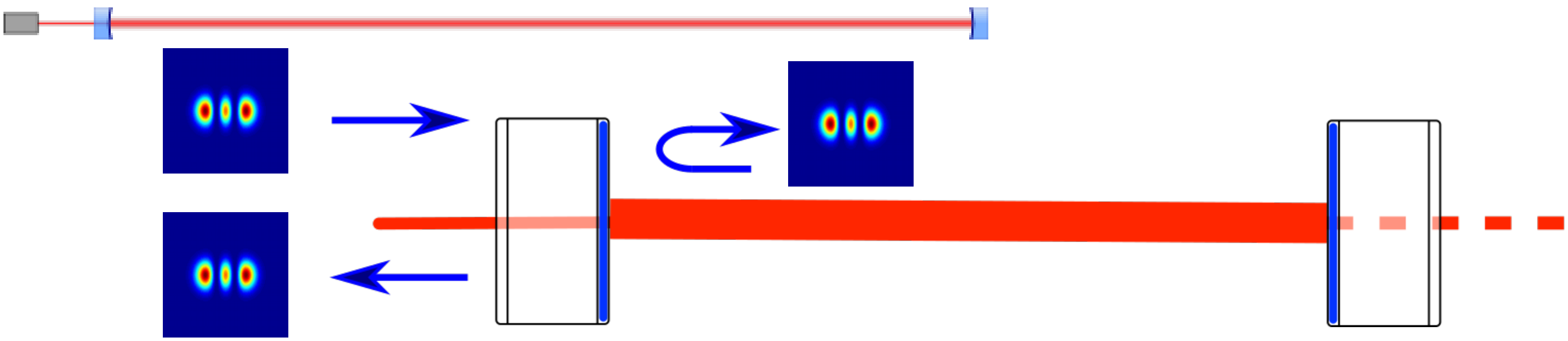
Eigen mode of FP cavity with spherical mirrors are the same as free space

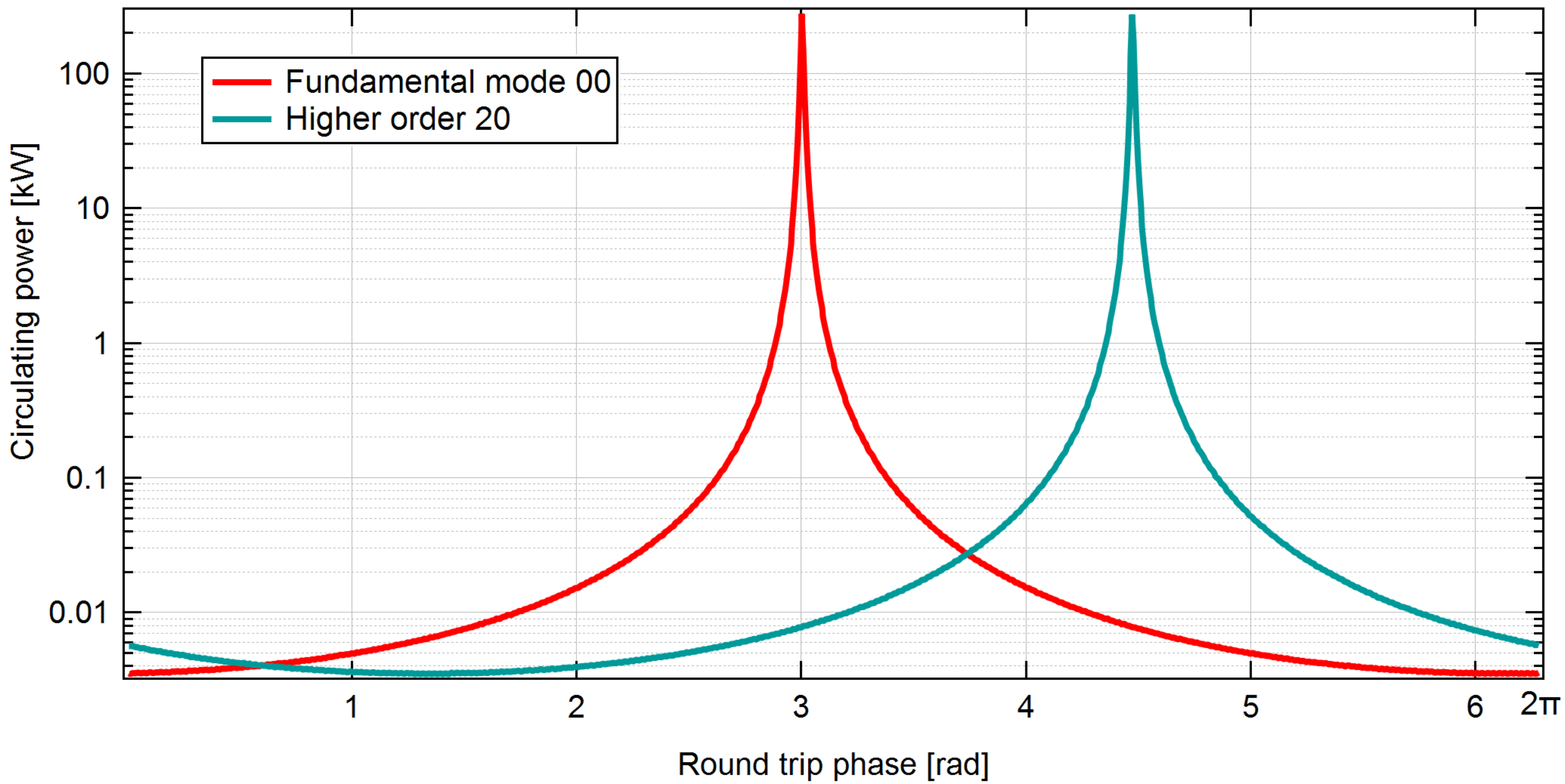
The optimal case



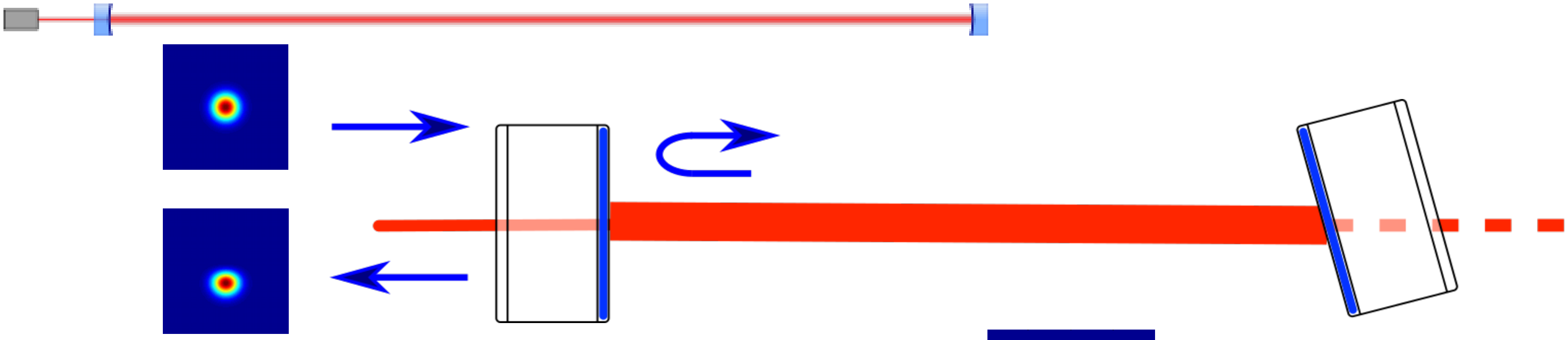


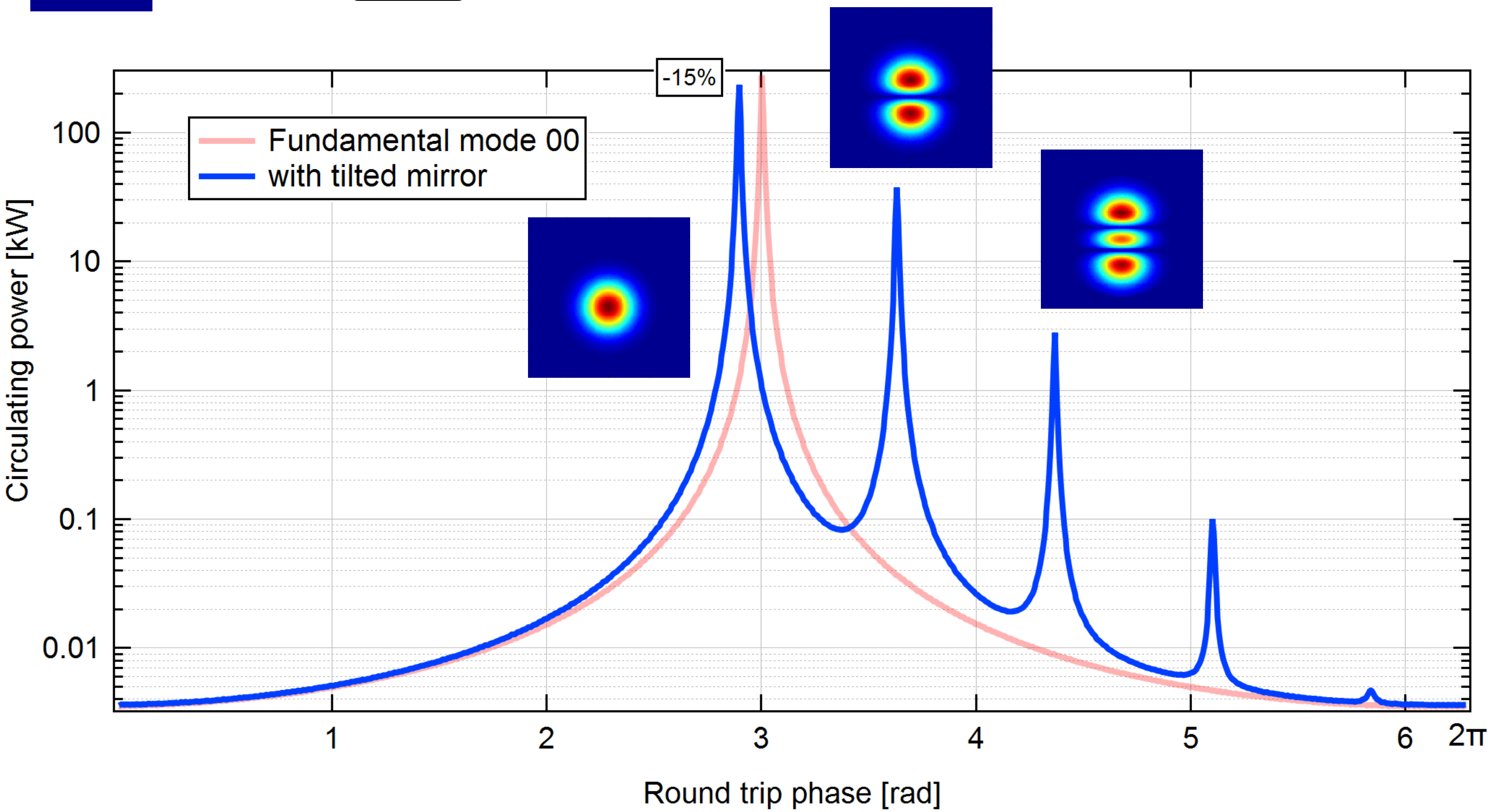
The optimal case



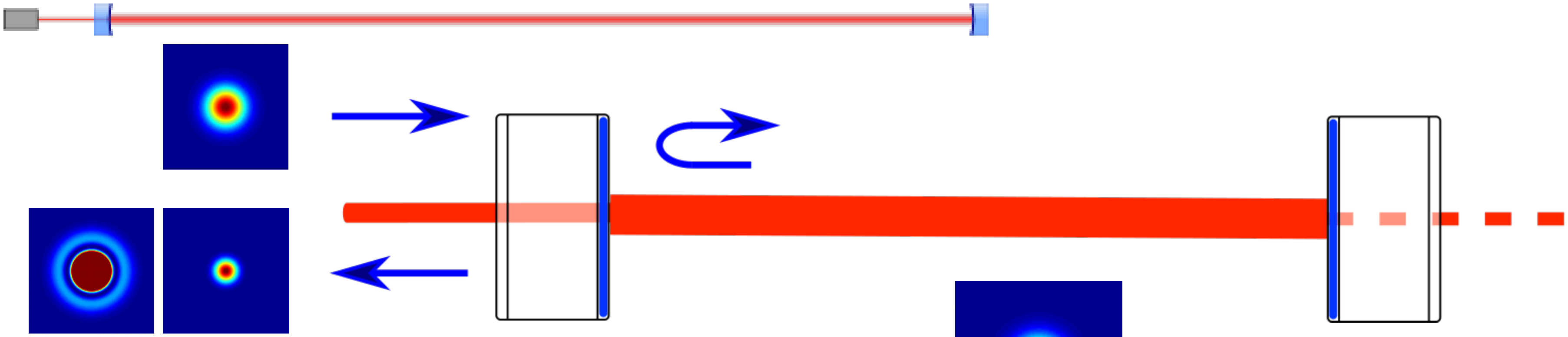


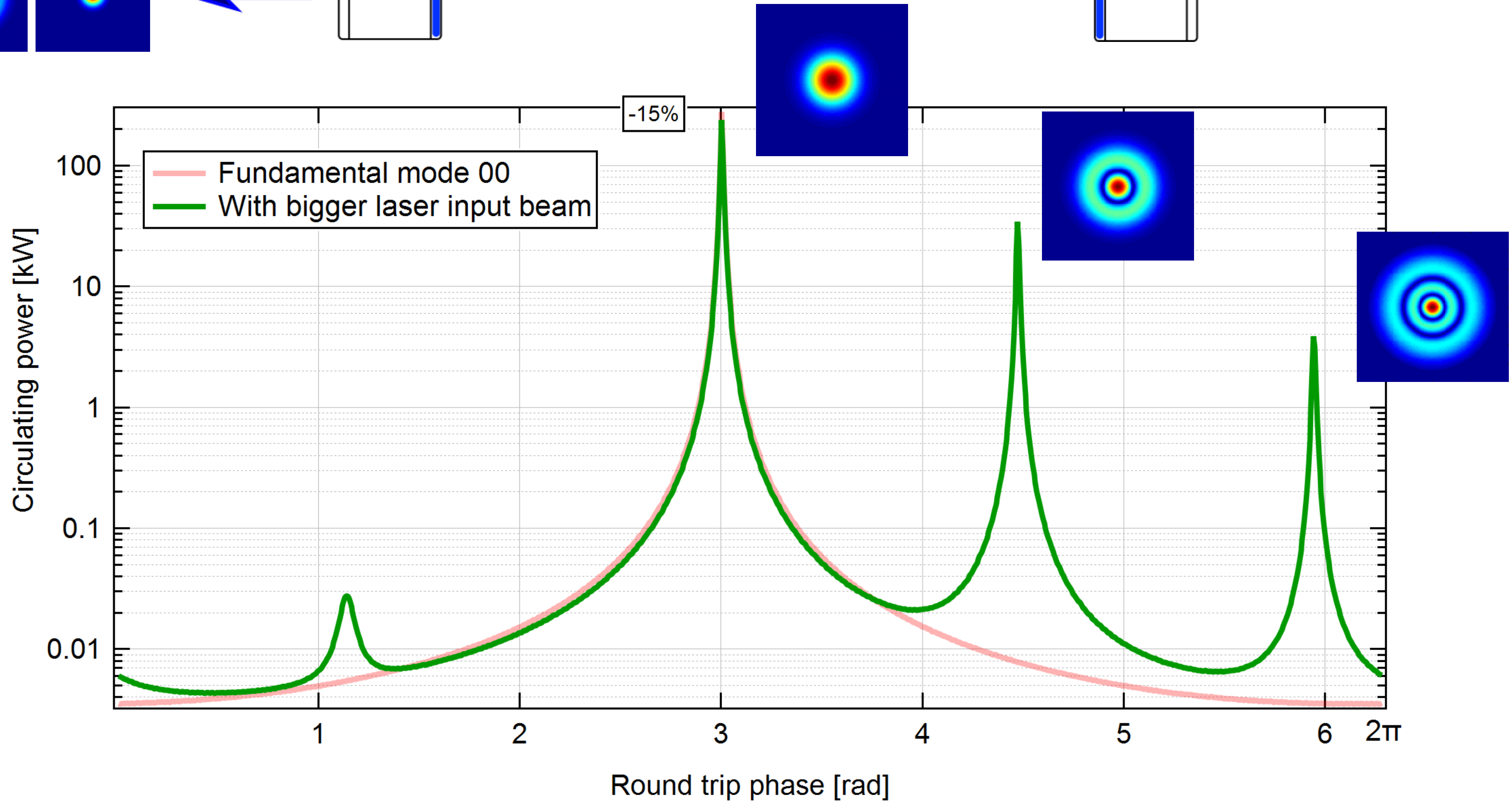
If I tilt the end mirror



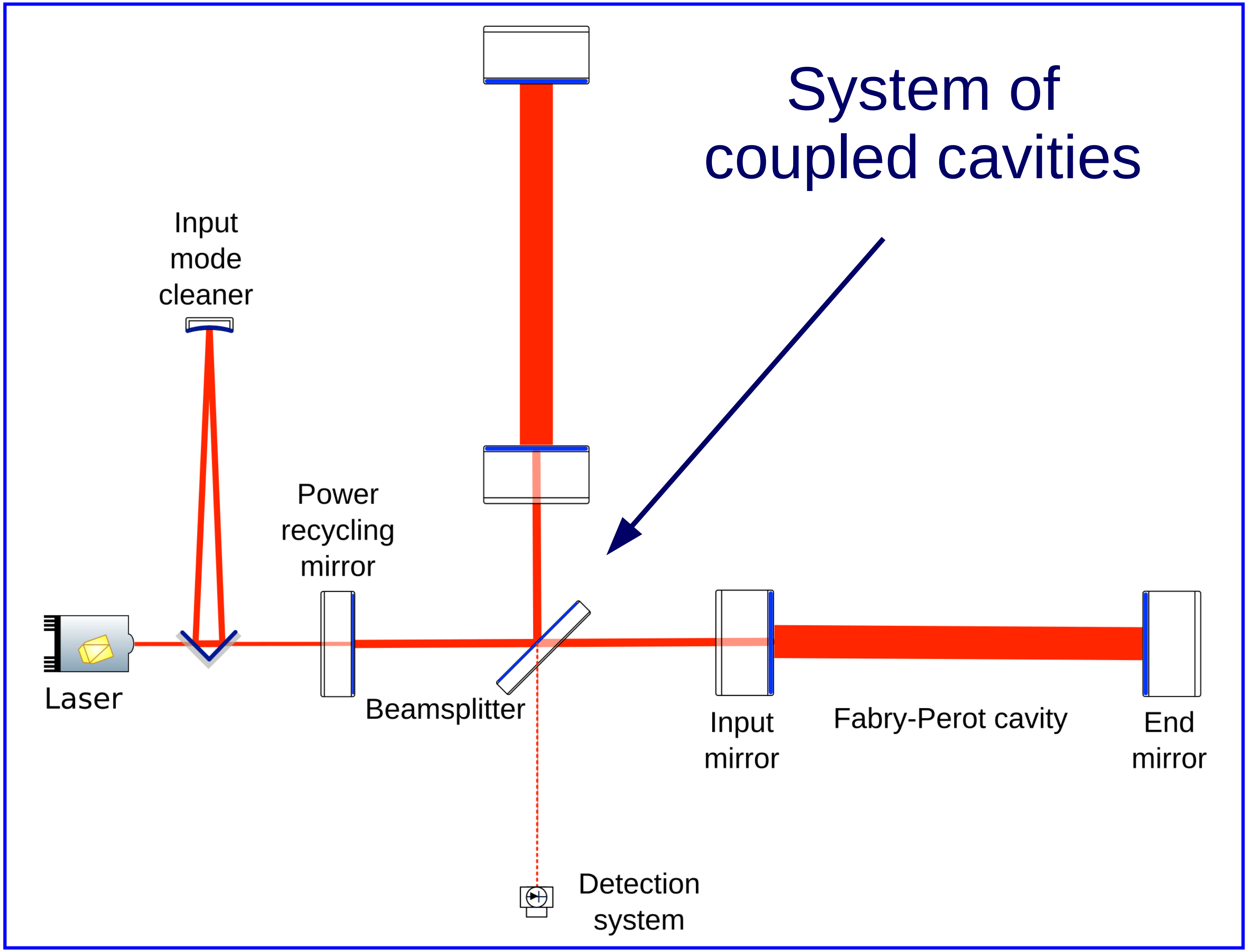


With a bigger input beam





For the interferometer



Misalignment / mode mismatching decreases the performances

To be continued,,,