# Application of the numerical conformal bootstrap

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Bootstat 2021

# Conformal Bootstrap

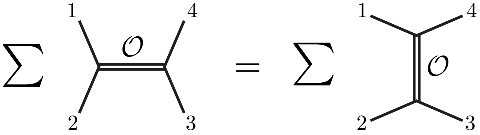
Can we use:

1. Conformal Symmetry: $SO(D,2)$ or $SO(D+1,1)$
2. Crossing Symmetry
3. Unitarity or Reflection Positivity

to map out and solve conformal field theories?

# Crossing Symmetry

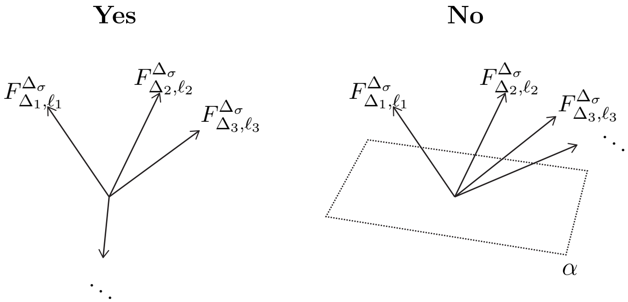
CFT 4-point functions $\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle$ can be expanded using OPE $\sigma\times\sigma\sim\sum_{\mathcal{O}}\lambda_{\sigma\sigma\mathcal{O}}\mathcal{O}$ in different channels, giving crossing constraint:



$$0 \;=\; \sum_{\Delta,\ell}\lambda^2_{\sigma\sigma\mathcal{O}}\left[\frac{g_{\Delta,\ell}(z,\overline{z})}{(z\overline{z})^{\Delta_\sigma}}-\frac{g_{\Delta,\ell}(1-z,1-\overline{z})}{[(1-z)(1-\overline{z})]^{\Delta_\sigma}}\right]$$

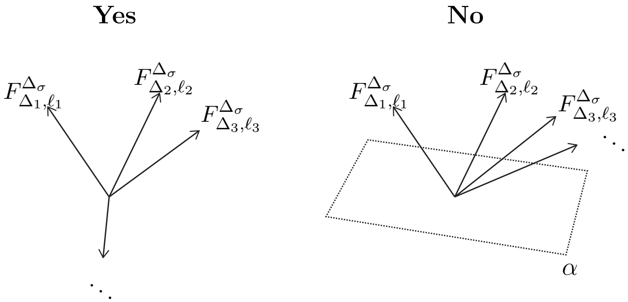
- Conformal blocks $g_{\Delta,\ell}(z,\overline{z})$ known functions of cross ratios $z,\overline{z}$
- Only unknowns are set of scaling dimensions and coefficents: $\{\Delta,\lambda_{ijk}\}$

# Numerical Approach



- Make some assumption on $\{\Delta, \lambda_{ijk}\}$, search for functional $\alpha = \left( \sum_{m+n\leq\Lambda} \alpha_{mn} \partial_z^m \partial_{\bar{z}}^n \Big|_{1/2,1/2} \right)$ implying $0 = \sum$(positive)

# Numerical Approach



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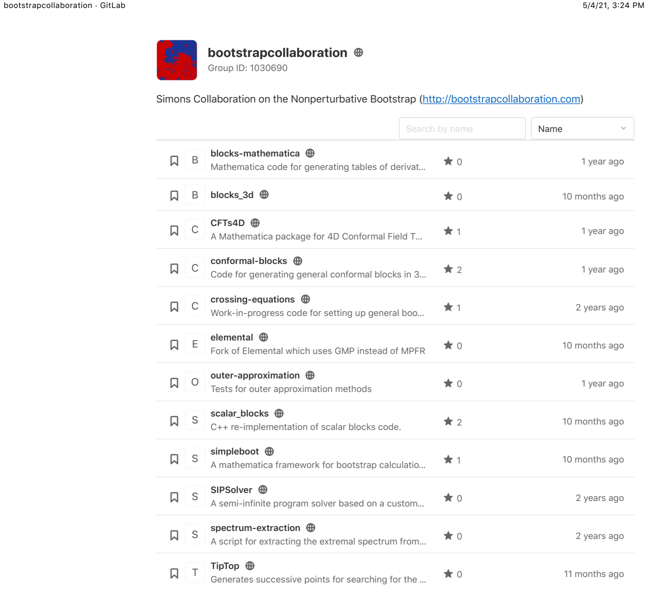
- Find them by solving semidefinite programs: `SDPB` https://github.com/davidsd/sdpb [Simmons-Duffin '15; Landry, Simmons-Duffin '19]

# Recent Advances

A few advances in the numerical bootstrap in the past 1-2 years:

- First applications of `SDPB 2.0` [Landry, Simmons-Duffin '19]
- New "Cutting Surface" algorithm for finding allowed $\lambda$'s in large parameter spaces [Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '19]
- New `tiptop` algorithm for maximizing some $\Delta$ over multi-dimensional island [Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi, '20]
- New `blocks_3d` software for efficient computation of spinning 3d conformal blocks [Erramilli, Iliesiu, Kravchuk, Landry, DP, Simmons-Duffin '20]

# http://gitlab.com/bootstrapcollaboration



bootstrapcollaboration · GitLab 5/4/21, 3:24 PM

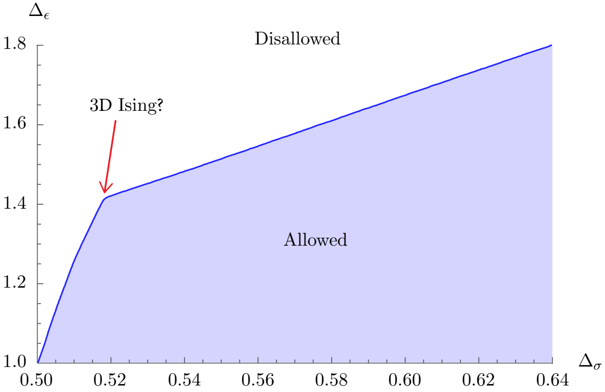
**bootstrapcollaboration**

Group ID: 1030690

Simons Collaboration on the Nonperturbative Bootstrap (http://bootstrapcollaboration.com)

| | | | |
|---|---|---|---|
| B | **blocks-mathematica**<br>Mathematica code for generating tables of derivat... | ★ 0 | 1 year ago |
| B | **blocks_3d** | ★ 0 | 10 months ago |
| C | **CFTs4D**<br>A Mathematica package for 4D Conformal Field T... | ★ 1 | 1 year ago |
| C | **conformal-blocks**<br>Code for generating general conformal blocks in 3... | ★ 2 | 1 year ago |
| C | **crossing-equations**<br>Work-in-progress code for setting up general boo... | ★ 1 | 2 years ago |
| E | **elemental**<br>Fork of Elemental which uses GMP instead of MPFR | ★ 0 | 10 months ago |
| O | **outer-approximation**<br>Tests for outer approximation methods | ★ 0 | 1 year ago |
| S | **scalar_blocks**<br>C++ re-implementation of scalar blocks code. | ★ 2 | 10 months ago |
| S | **simpleboot**<br>A mathematica framework for bootstrap calculatio... | ★ 1 | 10 months ago |
| S | **SIPSolver**<br>A semi-infinite program solver based on a custom... | ★ 0 | 2 years ago |
| S | **spectrum-extraction**<br>A script for extracting the extremal spectrum from... | ★ 0 | 2 years ago |
| T | **TipTop**<br>Generates successive points for searching for the ... | ★ 0 | 11 months ago |

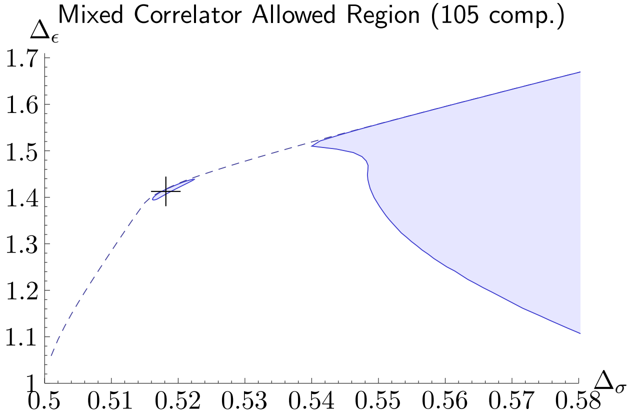
# 3D Dimension Bounds



[El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi, '12; '14]

- Upper bound on first $\mathbb{Z}_2$-even scalar in $\sigma \times \sigma \sim \mathbb{1} + \epsilon + \ldots$ from $\langle \sigma\sigma\sigma\sigma \rangle$

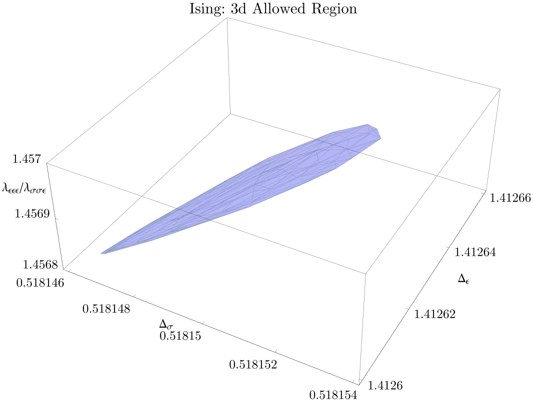
# 3D Ising Island



[Kos, DP, Simmons-Duffin '14]

- Combine 5 crossing relations from $\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle$ and impose that $\sigma$ and $\epsilon$ are the only relevant ($\Delta < 3$) operators

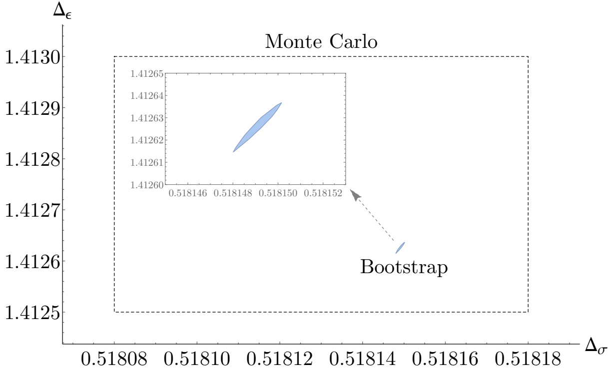
# Mixed Correlator Islands



[Kos, DP, Simmons-Duffin, Vichi '16]

- Best bounds: perform "OPE scan" over ratio $r \equiv \lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon} \rightarrow$ 3d island
- Excludes degenerate exchanged operators at same $\Delta_{\sigma,\epsilon}$ but different $\lambda$'s
- Each run imposes positivity in specific direction: $(1\ r)\vec{\alpha}\cdot\vec{F}_{2\times 2}(1\ r)^T > 0$

# 3D Ising Island



[Kos, DP, Simmons-Duffin, Vichi '16]

- Increase search space to $5 \times 253 = 1265$ components ($\Lambda = 43$)

$$\begin{aligned}\{\Delta_\sigma, \Delta_\epsilon\} &= \{0.518149(1), 1.412625(10)\}\\ \{\lambda_{\sigma\sigma\epsilon}, \lambda_{\epsilon\epsilon\epsilon}\} &= \{1.0518537(41), 1.532435(19)\}\end{aligned}$$

# 3D O(N) Models

Generalization to $N$ scalars $\phi_i$: $\mathcal{L}^{int}_{O(N)} \sim \lambda(\phi_i\phi^i)^2$

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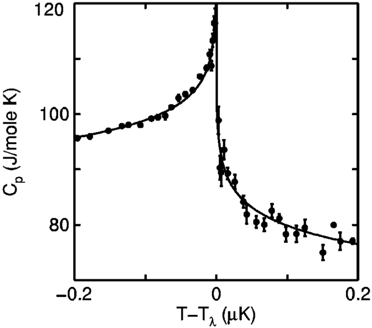

- $N=2$: Superfluid ($\lambda$) transition in $^4$He [Lipa et al, '96; '03]

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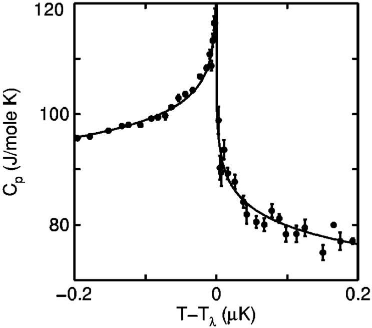
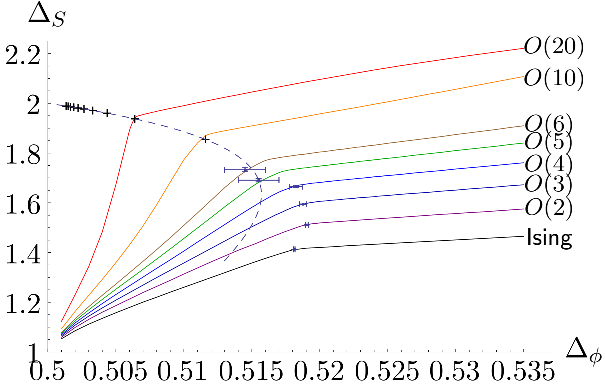


- $N = 2$: Superfluid ($\lambda$) transition in $^4$He [Lipa et al, '96; '03]
- $N = 3$: Isotropic ferromagnets (Fe, Co, Ni, ...)
- Large $N$: Solvable in $1/N$ expansion

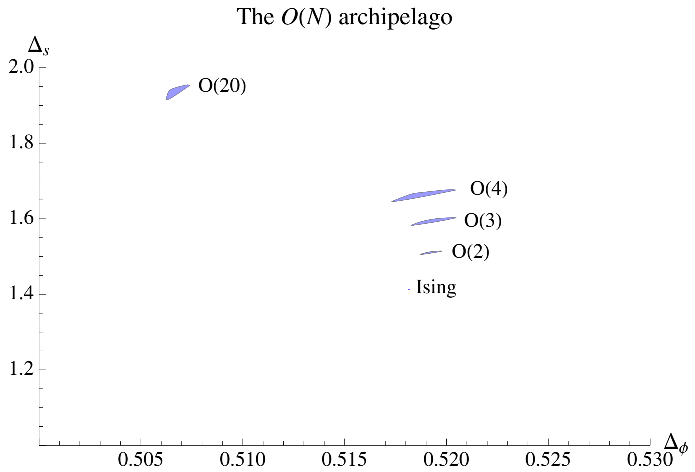
# 3D $O(N)$ Bounds



[Kos, DP, Simmons-Duffin '13]

- Extension to $\langle \phi_i \phi_j \phi_k \phi_l \rangle$, where $\phi_i$ is $O(N)$ vector
- Large $N$: matches $1/N$ expansion, Small $N$: matches experiment!

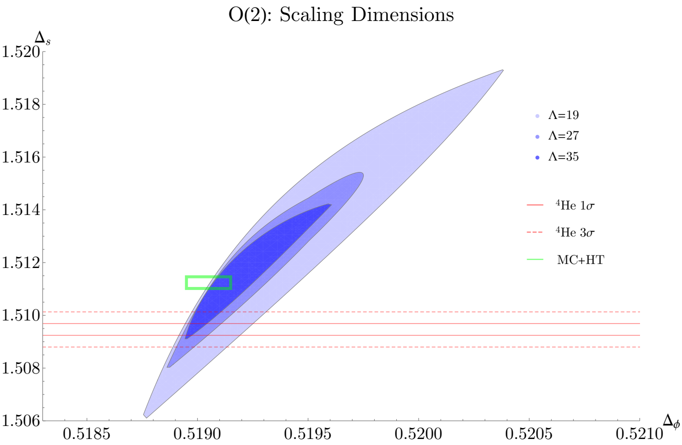
# $O(N)$ Archipelago from Mixed Correlators



[Kos, DP, Simmons-Duffin, Vichi '15; '16]

- With $\{\phi_i, s\}$ correlators, again find islands (Input: $\Delta_{\phi'} > 3, \Delta_s > 3$)

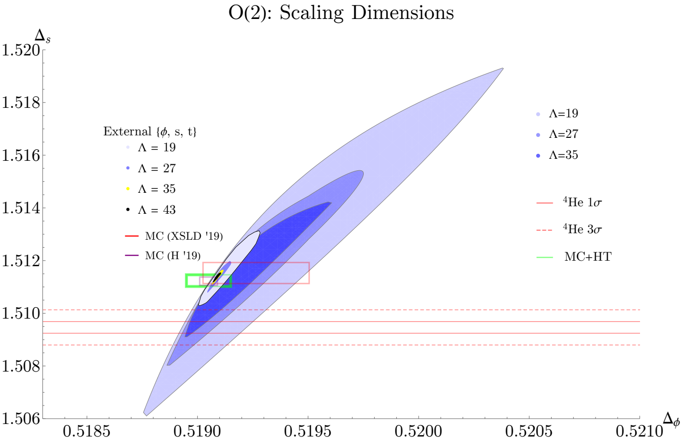
# $O(2)$ from $\{\phi_i, s\}$ System



[Kos, DP, Simmons-Duffin, Vichi '15; '16]

- $\{\Delta_\phi, \Delta_s, \lambda_{\phi\phi s}, \lambda_{sss}\} = \{.51926(32), 1.5117(25), .68726(65), .8286(60)\}$
- $8\sigma$ discrepancy between lattice and expt ($^4$He)

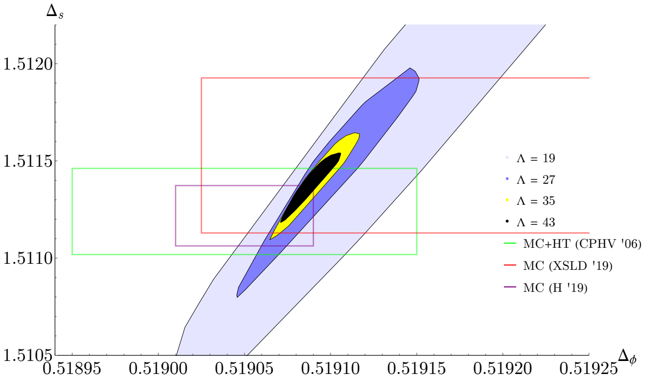
# $O(2)$ from $\{\phi_i, s, t_{ij}\}$ System



[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '19]

- Result from $\{\phi_i, s, t_{ij}\}$ system (22 crossing equations)

# $O(2)$ from $\{\phi_i, s, t_{ij}\}$ System



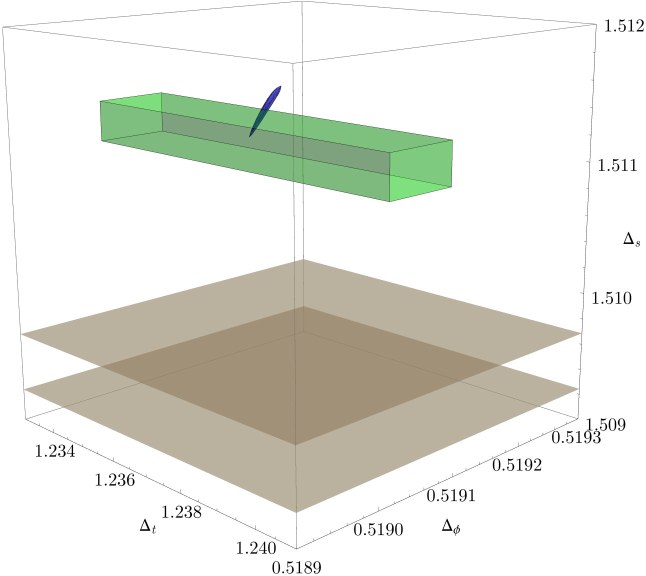
[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '19]

Bootstrap: $\alpha = -0.01527(21)$

MC: $\alpha = -0.01507(21)$ [Hasenbusch '19]

$^4$He: $\alpha = -0.0127(3)$ [Lipa et al, '96; '03]

# $O(2)$ from $\{\phi_i, s, t_{ij}\}$ System



[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '19]

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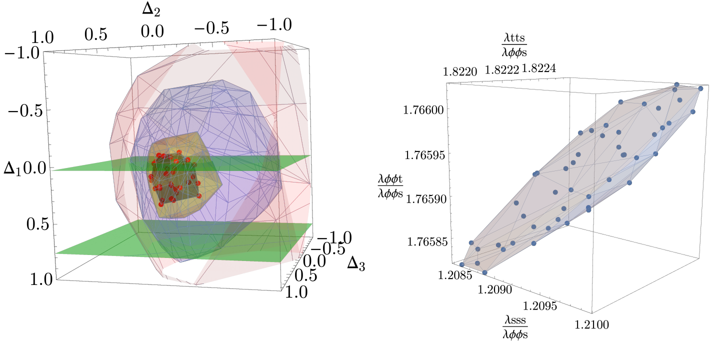


[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi, '19]

- Best results require 6d search over $\{\Delta_\phi, \Delta_s, \Delta_t, \frac{\lambda_{sss}}{\lambda_{\phi\phi s}}, \frac{\lambda_{tts}}{\lambda_{\phi\phi s}}, \frac{\lambda_{\phi\phi t}}{\lambda_{\phi\phi s}}\}$

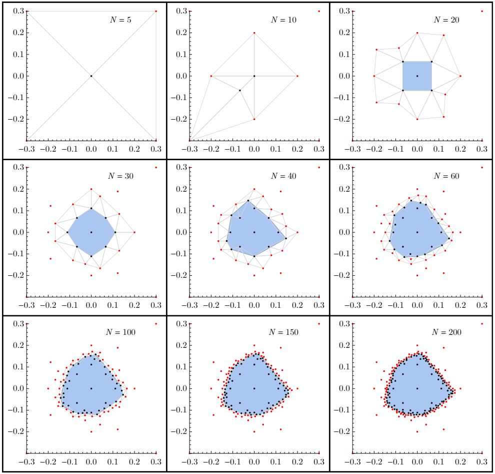
$$\begin{aligned}\{\Delta_\phi, \Delta_s, \Delta_t\} &= \{0.519088(17), 1.51136(18), 1.23629(9)\}\\ \{\frac{\lambda_{sss}}{\lambda_{\phi\phi s}}, \frac{\lambda_{tts}}{\lambda_{\phi\phi s}}, \frac{\lambda_{\phi\phi t}}{\lambda_{\phi\phi s}}\} &= \{1.20926(46), 1.82227(19), 1.765918(64)\}\end{aligned}$$

# $O(2)$ from $\{\phi_i, s, t_{ij}\}$ System

To carry out this 6d search, we employed the following strategy:

1. Use `SDPB 2.0` [Landry, Simmons-Duffin '19], take advantage of parallelization
2. Use hotstarting [Go, Tachikawa '19] to run `SDPB` for fewer iterations
3. Search over $\Delta$'s carried out using "Delaunay triangulation" search
4. Search over $\lambda$'s carried out using "Cutting Surface" algorithm

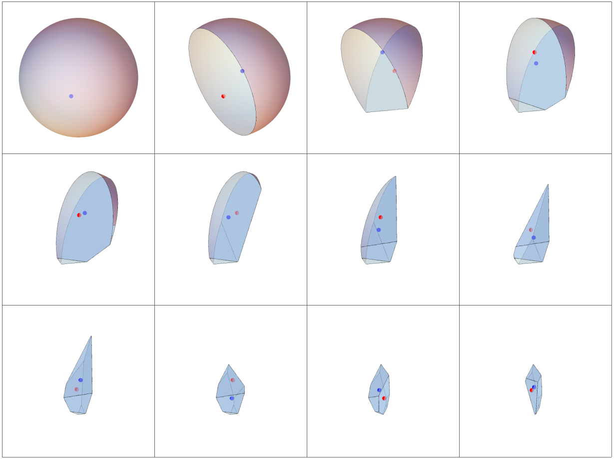
# Delaunay Triangulation Search



[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '19]

- Compute Delaunay triangulation of all tested points, pick midpoint of "biggest" triangle connecting disallowed to allowed

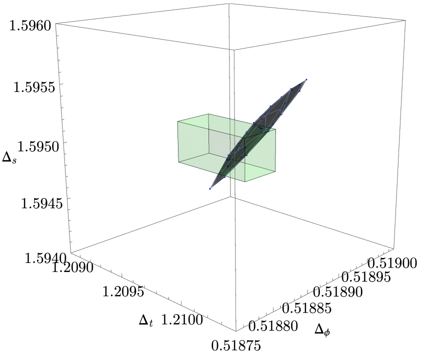
# Cutting Surface Algorithm



[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '19]

- Each computation excludes a *region* of $\vec{\lambda}$-space: $\vec{\lambda} \cdot \alpha[F_{\vec{\Delta}}] \cdot \vec{\lambda} > 0$
- After $\sim 10 - 30$ tests either find allowed point or rule out entire region

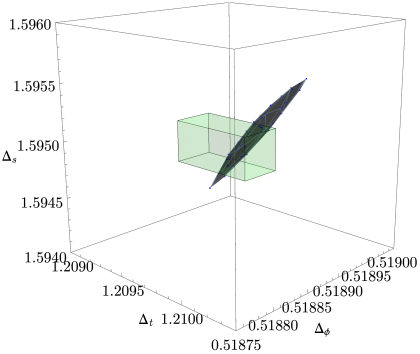
# $O(3)$ from $\{\phi_i, s, t_{ij}\}$ System



Black (Bootstrap): [Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi '20]
Green (Monte Carlo): [Hasenbusch, Vicari '11; Hasenbusch '20]

- New $O(3)$ island: $\{\Delta_\phi, \Delta_s, \Delta_t\} = \{0.51894(5), 1.5949(6), 1.2095(2)\}$
- Open question: is $\phi^{\{i}\phi^j\phi^k\phi^{l\}}$ relevant or irrelevant in $O(3)$ model?

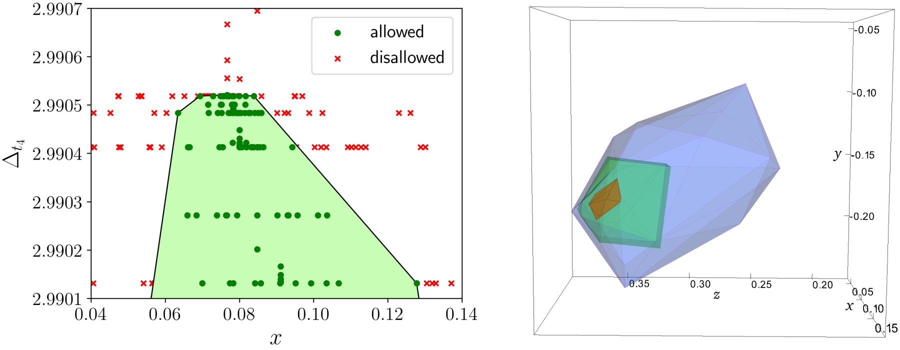
# $O(3)$ from $\{\phi_i, s, t_{ij}\}$ System



Black (Bootstrap): [Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi, '20]
Green (Monte Carlo): [Hasenbusch, Vicari '11; Hasenbusch '20]

- Using `tiptop` search, find it is relevant: $\Delta_{\phi^{\{i}\phi^{j}\phi^{k}\phi^{l\}}} < 2.99056$!
- Proof that critical Heisenberg magnets are unstable to cubic anisotropy, should flow to fixed point with cubic symmetry $C_3$ rather than $O(3)$

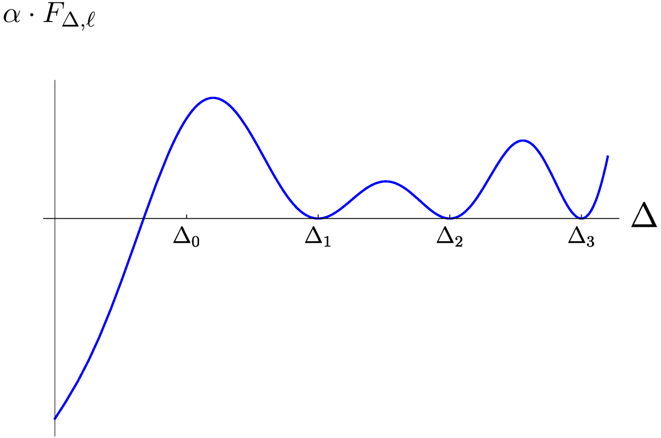
# $O(3)$ from $\{\phi_i, s, t_{ij}\}$ System



[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi, '20]

- `tiptop` search: map out approximate island at a given gap $\Delta_{t_4} - 3$, shrink bounding box of island by factor of 2, increase gap (via binary search) until it can no longer accommodate smaller box, then iterate

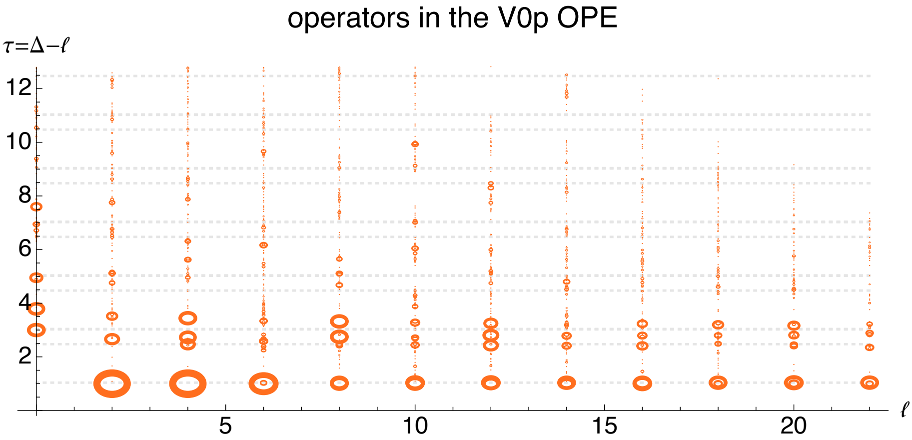
# Extremal Functional



[DP, Simmons-Duffin '10; Paulos, El-Showk '12; plot from Paulos, Zan '20]

- By going to a boundary of the allowed region (e.g., extremizing $\lambda_{\phi\phi s}$), we can extract extremal spectra corresponding to the zeros of $\alpha \cdot F_{\Delta,\ell}$

# $O(2)$ from $\{\phi_i, s, t_{ij}\}$ System



[Chester, Landry, Liu, DP, Simmons-Duffin, Su, Vichi, '19]

- Can be done in practice using `spectrum-extraction` Python script (`spectrum.py`), which uses `sdpb` output

# Analytical Bootstrap

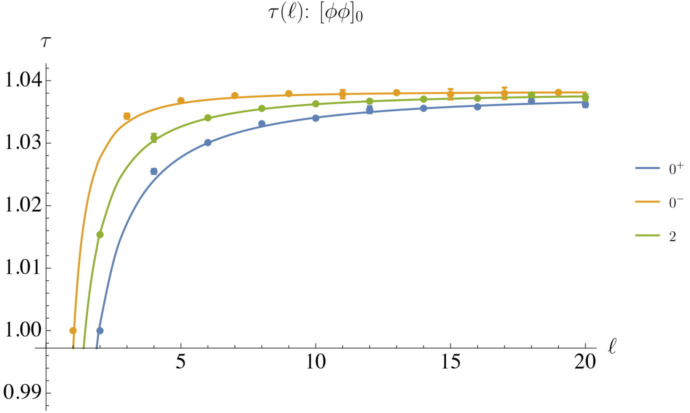
Extremal spectra can be compared with analytical bootstrap predictions:

- Lightcone Bootstrap ($z \to 0$): $\exists$ trajectories of "double-twist" operators $\sim \sigma \partial^\ell \sigma$ with twist asymptoting to $\tau(\ell \to \infty) = 2\Delta_\sigma - \frac{\#}{\ell} - \frac{\#}{\ell^{\Delta_\epsilon}} + \dots$
  [Fitzpatrick, Kaplan, DP, Simmons-Duffin '12; Komargodski, Zhiboedov '12]

- Lorentzian Inversion → All-orders analytic function [Caron-Huot '17]

$$\tau(\ell) \sim \int \mathrm{dDisc}[g] \sim \sum_{\mathcal{O}} {}_4F_3(\dots)$$ [see Henriksson talk...]

# $O(2)$: Comparison with Analytics



[Albayrak, Meltzer, DP '19; J. Liu, Meltzer, DP, Simmons-Duffin '20]

- Excellent agreement between leading-twist extremal spectra and analytics after including exchange of $\{s, t, J^\mu, T^{\mu\nu}\}$

# The Spinning Frontier

# `blocks_3d` Software

- General 3d conformal blocks can be expanded recursively in poles:

$$g^{ab}_{\Delta,j,I} \sim \frac{1}{\Delta - \Delta_{j,i}} (\mathcal{L}_{j,i})^{a}_{a'} (\mathcal{R}_{j,i})^{b}_{b'} g^{a'b'}_{\Delta'_{j,i}, j'_{j,i}, I}(z, \bar{z})$$

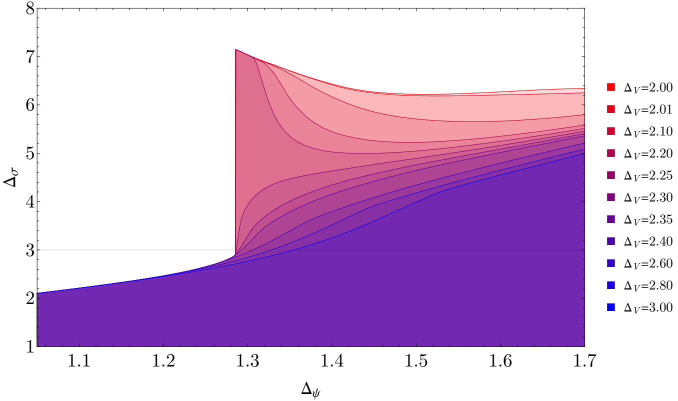
- `blocks_3d` is efficient, multithreaded, C++ implementation

  [Erramilli, Iliesiu, Kravchuk '19; Erramilli, Iliesiu, Kravchuk, Landry, DP, Simmons-Duffin '20]
- Practical for external fermions $\psi$, currents $J$, stress-tensors $T$, ...

| block ($\Lambda = 25$) | $j_{12}$ | $j_{43}$ | Memory (GB) | Time (hr) |
|---|---|---|---|---|
| $\langle\phi\phi\phi\phi\rangle$ | 0 | 0 | 4 | 0.014 |
| $\langle\phi\psi\phi\psi\rangle$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 7 | 0.025 |
| $\langle T\phi\phi\phi\rangle$ | 2 | 0 | 11 | 0.045 |
| $\langle\psi\psi\psi\psi\rangle$ | 1 | 1 | 15 | 0.068 |
| $\langle T\phi T\phi\rangle$ | 2 | 2 | 36 | 0.20 |
| $\langle T\psi T\psi\rangle$ | $\frac{5}{2}$ | $\frac{5}{2}$ | 48 | 0.62 |
| $\langle TTT\phi\rangle$ | 4 | 2 | 62 | 0.94 |
| $\langle TTTT\rangle$ | 4 | 4 | 106 | 6.9 |

(See `CFTs4D` package for spinning 4d blocks [Cuomo, Karateev, Kravchuk '17])

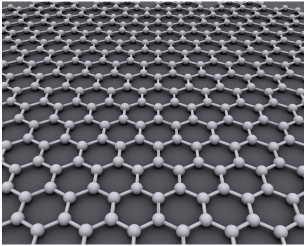
# Mysterious jump?



[Erramilli, Iliesiu, Kravchuk, Landry, DP, Simmons-Duffin '20]

- Sharp jump in parity-odd scalar bound from $\langle\psi\psi\psi\psi\rangle$
  [Iliesiu, Kos, DP, Pufu, Simmons-Duffin '17]
- Seems to persist after removing "fake primary effect"
  ([Karateev, Kravchuk, Serone, Vichi '19]: spin-1 $V^\mu$ mimics $\Delta = 3$ scalar)
- Could be evidence for new fermionic CFT w/ no relevant scalars?

# 3D Fermion Models (Gross-Neveu-Yukawa)



Interesting CFTs obtained from fixed points involving $N$ fermions:

$\mathcal{L}_{GNY} \sim \frac{g}{2}\sigma\overline{\psi}^i\psi_i + \lambda\sigma^4$ (and variations with multiple scalars)

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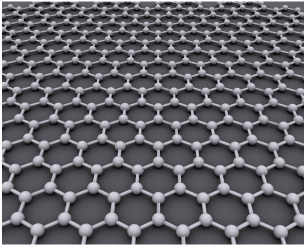
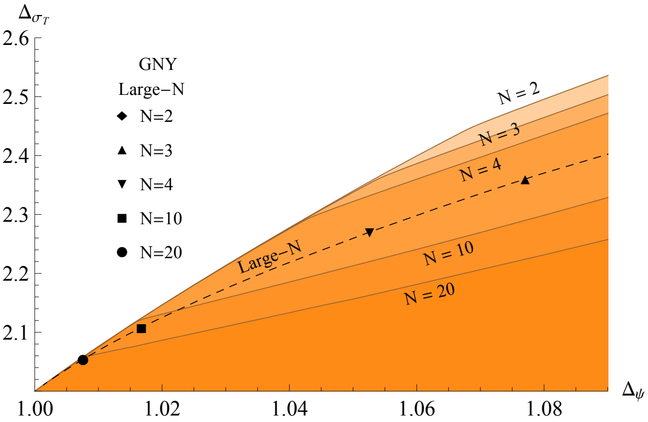


Interesting CFTs obtained from fixed points involving $N$ fermions:

$\mathcal{L}_{GNY} \sim \frac{g}{2}\sigma\overline{\psi}^i\psi_i + \lambda\sigma^4$ (and variations with multiple scalars)

- Large $N$: Solvable in $1/N$ expansion [Gracey '92, '93; ...]
- $N = 8$: Possible QCPs in D-wave superconductors or graphene [Vojta, Zhang, Sachdev '00; Herbut '06; Classen, Herbut, Scherer '17]
- $N = 4$: Spinless fermions on honeycomb lattice, gapless semiconductors [Raghu, Qi, Honerkamp, Zhang '07; Moon, Xu, Kim, Balents '12; Herbut, Janssen '14]

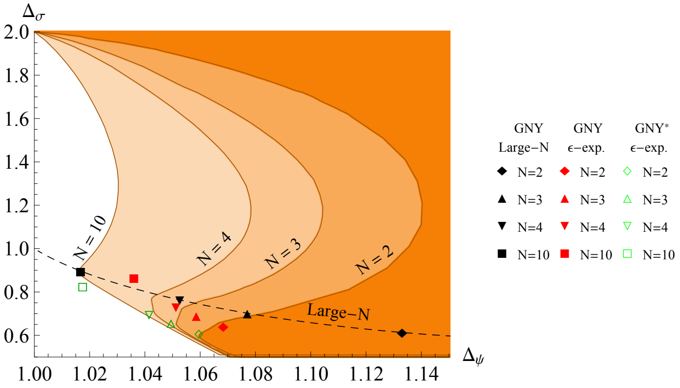
# 3D O(N) Fermion Bootstrap



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin '17]

- Bootstrap for fermion 4-point functions $\langle\psi_i\psi_j\psi_k\psi_l\rangle$
- Kinks in symmetric tensor bounds match GNY models at large $N$

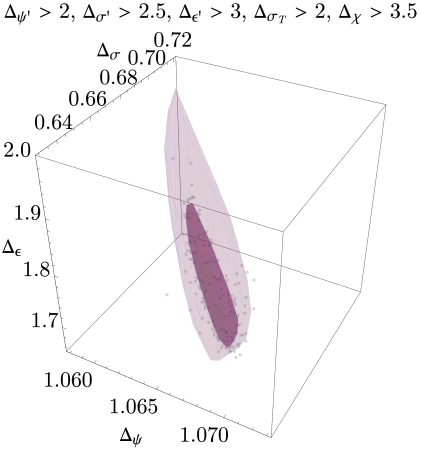
# 3D O(N) Fermion Bootstrap



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin '17]

- Intricate structure in $\{\Delta_\psi, \Delta_\sigma\}$ plane assuming $\sigma'$ irrelevant
- Upper kinks plausibly related to GNY models

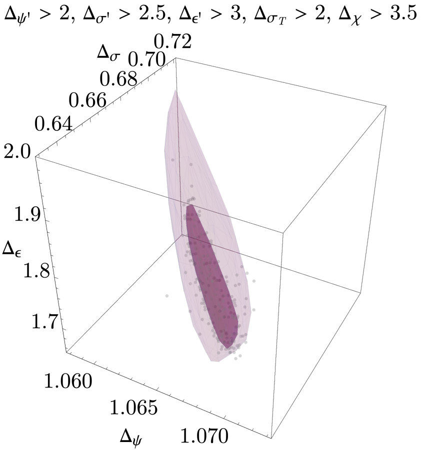
# Preliminary Island for $N = 2$ Gross-Neveu-Yukawa Model



[Erramilli, Iliesiu, Kravchuk, A. Liu, DP, Simmons-Duffin, in progress]

- Preliminary island from system containing $\{\sigma, \psi_i, \epsilon\}$ at $\Lambda = 11, 15$
- Gaps motivated by large $N$ estimates and E.O.M. $\partial\psi \sim \sigma\psi$

# Preliminary Island for $N = 2$ Gross-Neveu-Yukawa Model

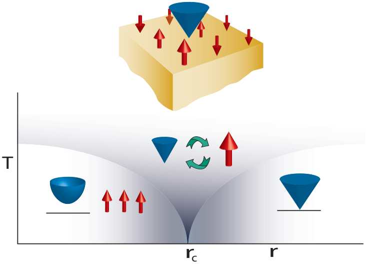


[Erramilli, Iliesiu, Kravchuk, A. Liu, DP, Simmons-Duffin, in progress]

$$\{\Delta_\psi, \Delta_\sigma, \Delta_\epsilon\} = \{1.066(4), 0.666(18), 1.77(7)\}$$

- Now preparing to run at high derivative order and at other $N$...

# Minimal 3D SCFT ($N = 1$ Gross-Neveu-Yukawa)



- The minimal SUSY extension of 3D Ising has $\mathcal{N} = 1$ supersymmetry:

$$V = \frac{g}{2}\sigma\overline{\psi}\psi + \frac{g}{8}\sigma^4 \qquad \leftrightarrow \qquad W = \frac{g}{3}\Sigma^3, \quad \Sigma = \sigma + \theta\psi + \theta^2\epsilon$$

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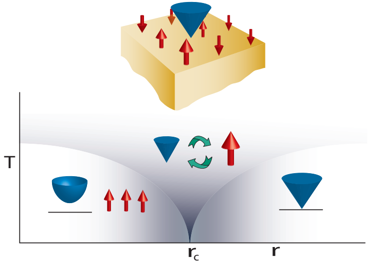

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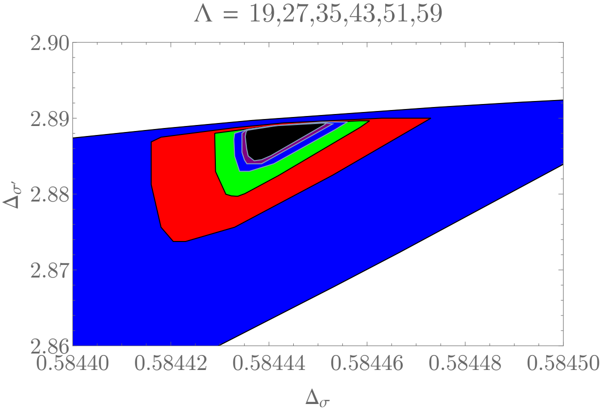
- May be realizable in (3+1)D topological superconductors, with (2+1)D boundary supporting Majorana fermions [Grover, Sheng, Vishwanath '13]

# Supersymmetric Island

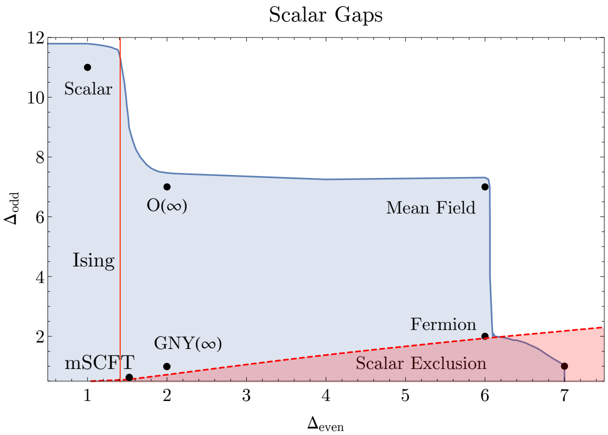


[Rong, Su '18; Atanasov, Hillman, DP, Rong, Su, in progress]

- $\{\sigma, \epsilon\}$ SUSY system $\rightarrow \Delta_\sigma = .584443(8)$, $\Delta_{\sigma'} = 2.8869(25)$
  Compare to $\epsilon$-expansion: $\Delta_\sigma = .5837(14)$ [Ihrig, Mihaila, Scherer '18]

(Assumption: $\mathcal{N} = 1$ SUSY, $\Delta_{\epsilon'} \geq 3$)

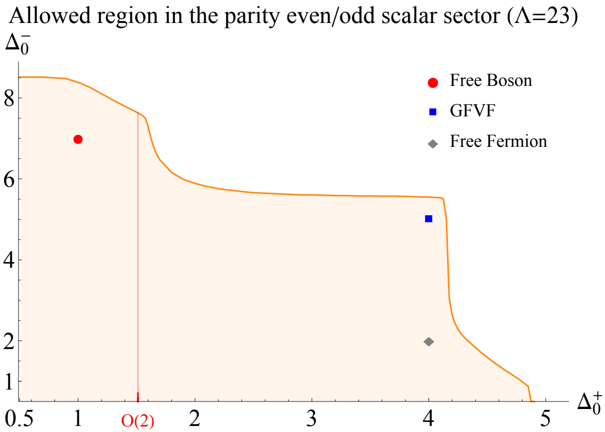
# Map of Allowed Scalar Gaps from $\langle T^{\mu\nu}T^{\rho\sigma}T^{\alpha\beta}T^{\gamma\delta}\rangle$



[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, '17]

- Allowed {scalar, pseudoscalar} gaps from stress tensor 4-point functions

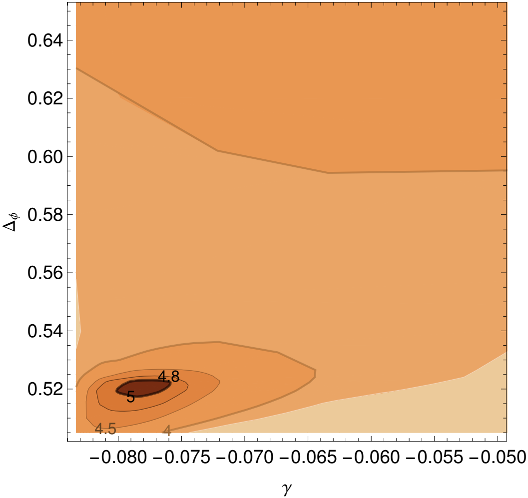
# Map of Allowed Scalar Gaps from $\langle J^\mu J^\nu J^\rho J^\sigma \rangle$



[Dymarsky, Penedones, Trevisani, Vichi '17]

- Allowed {scalar, pseudoscalar} gaps from current 4-point functions

# Map of Allowed Couplings from $\{J^\mu, \phi\}$ System



[Reehorst, Trevisani, Vichi '19]

- Allowed couplings $\langle JJT \rangle \propto \gamma$ in $O(2)$ model after imposing $T'_{\mu\nu}$ gap

# Bootstrap Future

Where do we go from here?

- Find islands for other interesting CFTs
  - 3d Gross-Neveu-Yukawa Models: $\mathcal{L} \sim \lambda \sigma \psi_i \psi^i$ + variations
  - Conformal windows of gauge theories (3d QED, 4d QCD, ...)
  - Superconformal zoo
  - Other interesting theories described in this workshop!

- Study larger systems of bootstrap equations
  - Mixed correlators with spinning operators ($\psi$, $J^\mu$, $T^{\mu\nu}$)
  - Improve algorithms and software tools

- Improve analytical understanding of bootstrap equations
  - Match to Lorentzian Inversion formula, conformal dispersion relations
  - Incorporate analytical insights into numerical algorithms

# Backup Slides

# Lorentzian Inversion Review

- The basic idea is to decompose CFT 4-pt functions $\langle\sigma\sigma\sigma\sigma\rangle \propto g(z,\overline{z})$ in a basis of "principal series" ($\Delta = d/2 + i\alpha$) partial waves

$$g(z,\overline{z}) = \sum_{\ell=0}^{\infty} \int_{d/2-i\infty}^{d/2+i\infty} \frac{d\Delta}{2\pi i} c(\Delta,\ell) g_{\Delta,\ell}(z,\overline{z}) + (\text{non-norm.})$$

  where the physical spectrum is encoded in the poles of $c(\Delta,\ell)$.

# Lorentzian Inversion Review

- Using orthogonality and Lorentzian continuation one "inverts" the formula to obtain $c(\Delta,\ell) = c^t(\Delta,\ell) + (-1)^\ell c^u(\Delta,\ell)$:

$$c^t(\Delta,\ell) \quad = \quad \frac{\kappa_{\Delta+\ell}}{4}\int_0^1 dz d\overline{z}\mu(z,\overline{z}) g_{\ell+d-1,\Delta+1-d}(z,\overline{z})\text{dDisc}\left[g(z,\overline{z})\right]$$

with

$$\begin{aligned} \mu(z,\overline{z}) \quad &= \quad \left|\frac{z-\overline{z}}{z\overline{z}}\right|^{d-2}\frac{1}{(z\overline{z})^2} \\ \text{dDisc}\left[g(z,\overline{z})\right] \quad &= \quad g(z,\overline{z}) - \frac{1}{2}g(z,\overline{z}e^{2\pi i}) - \frac{1}{2}g(z,\overline{z}e^{-2\pi i}) \end{aligned}$$

See [Caron-Huot '17; Simmons-Duffin, Stanford, Witten '17; Kravchuk, Simmons-Duffin '18]

# Lorentzian Inversion Review

- Expanding $g(z,\overline{z}) = \left(\frac{z\overline{z}}{(1-z)(1-\overline{z})}\right)^{\Delta_\sigma} \sum \lambda^2_{\sigma\sigma\mathcal{O}} g_{\Delta,\ell}(1-z,1-\overline{z})$ in a finite number of known contributions, we can compute the integrals

- Matching identity operator reveals poles $\frac{1}{\Delta-(2\Delta_\sigma+\ell)}$ corresponding to "double-twist" operators: $\sigma\partial_{\mu_1}\ldots\partial_{\mu_\ell}\sigma$

- Other exchanged operators give anomalous dimensions ($\log(z)$ terms) and correct their OPE coefficients (regular terms)

Approach developed in various works: [Sleight, Taronna '18; Kravchuk, Simmons-Duffin '18; Cardona, Sen '18; Karateev, Kravchuk, Simmons-Duffin '18; Cardona, Guha, Kanumilli, Sen '18; Albayrak, Meltzer, DP '19, '20; Caron-Huot, Gobeil, Zahraee '20]

# Anomalous Dimensions from Scalar Exchange

$$\delta\tau_{[\sigma\sigma]_0}(\bar{h})_{\mathsf{pert}} = -\frac{\lambda^2_{\sigma\sigma\epsilon}}{1+\delta P_{[\sigma\sigma]_0}(\bar{h})}\frac{2\Gamma(\Delta_\sigma)^2\Gamma\left(1+\frac{\Delta_\epsilon-2\Delta_\sigma}{2}\right)^2\Gamma(\Delta_\epsilon)}{\Gamma\left(1-\frac{\Delta_\epsilon-2\Delta_\sigma}{2}\right)^2\Gamma\left(\frac{\Delta_\epsilon-2\Delta_\sigma}{2}\right)^2\Gamma\left(\frac{\Delta_\epsilon}{2}\right)^2}$$

$$\times\frac{\Gamma(\bar{h}-\Delta_\sigma+1)\Gamma\left(\bar{h}-\frac{\Delta_\epsilon-2\Delta_\sigma}{2}-1\right)}{\Gamma(\bar{h}+\Delta_\sigma-1)\Gamma\left(\bar{h}+\frac{\Delta_\epsilon-2\Delta_\sigma}{2}+1\right)}$$

$$\times\,{}_4F_3\left(\begin{matrix}\frac{\Delta_\epsilon-d+2}{2},\frac{\Delta_\epsilon-2\Delta_\sigma}{2}+1,\frac{\Delta_\epsilon-2\Delta_\sigma}{2}+1,\frac{\Delta_\epsilon}{2}\\ \frac{\Delta_\epsilon-2\Delta_\sigma}{2}-\bar{h}+2,\frac{\Delta_\epsilon-2\Delta_\sigma}{2}+\bar{h}+1,\Delta_\epsilon-\frac{d-2}{2}\end{matrix};1\right)$$

$$\delta\tau_{[\sigma\sigma]_0}(\bar{h})_{\mathsf{np}} = -\frac{\lambda^2_{\sigma\sigma\epsilon}}{1+\delta P_{[\phi\phi]_0}(\bar{h})}\frac{2\Gamma(\Delta_\sigma)^2\Gamma\left(\Delta_\epsilon-\frac{d-2}{2}\right)\Gamma(\Delta_\epsilon)}{\Gamma\left(1-\frac{\Delta_\epsilon-2\Delta_\sigma}{2}\right)^2\Gamma\left(\frac{\Delta_\epsilon-2\Delta_\sigma}{2}\right)^2\Gamma\left(\frac{\Delta_\epsilon}{2}\right)^3\Gamma\left(\frac{\Delta_\epsilon-d-2}{2}\right)}$$

$$\times\frac{\Gamma(\bar{h})^2\Gamma(\bar{h}-\Delta_\sigma+1)\Gamma\left(\bar{h}+\Delta_\sigma-\frac{d}{2}\right)\Gamma\left(\frac{\Delta_\epsilon-2\Delta_\sigma}{2}-\bar{h}+1\right)}{\Gamma(2\bar{h})\Gamma\left(\bar{h}+\Delta_\sigma+\frac{\Delta_\epsilon}{2}-\frac{d}{2}\right)}$$

$$\times\,{}_4F_3\left(\begin{matrix}\bar{h},\bar{h},\bar{h}+\Delta_\sigma-1,\bar{h}+\Delta_\sigma-\frac{d}{2}\\ 2\bar{h},\bar{h}-\frac{\Delta_\epsilon-2\Delta_\sigma}{2},\bar{h}+\Delta_\sigma+\frac{\Delta_\epsilon}{2}-\frac{d}{2}\end{matrix};1\right)$$

- E.g., for exchange of a scalar $\epsilon$, integral yields two pieces: $\delta\tau_{[\sigma\sigma]_0}(\bar{h}) = \delta\tau_{[\sigma\sigma]_0}(\bar{h})_{\mathsf{pert}} + \delta\tau_{[\sigma\sigma]_0}(\bar{h})_{\mathsf{np}}$ (here $\bar{h}\equiv\frac{\Delta+\ell}{2}$)