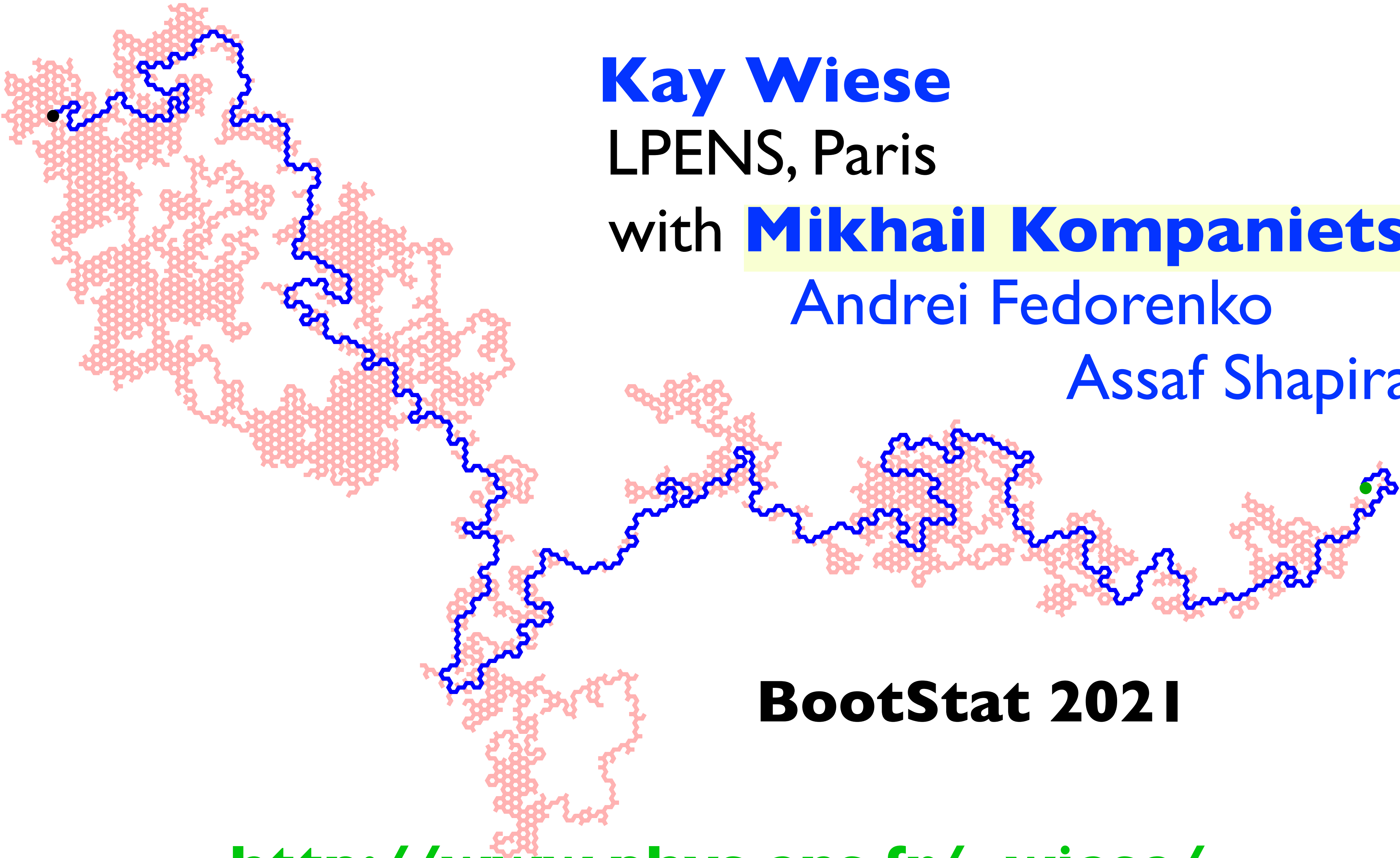
How can the perturbative renormalization group help the bootstrap, and what are interesting questions for the bootstrap?



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BootStat 2021

http://www.phys.ens.fr/~wiese/

A purposeful reminder on RG basics

$O(n)$ model: $$\mathcal{S}[\overrightarrow{\phi}] = \int_x \frac{1}{2}[\nabla \overrightarrow{\phi}(x)]^2 + \frac{m^2}{2}\overrightarrow{\phi}(x)^2 + \frac{g}{4}[\overrightarrow{\phi}(x)^2]^2$$

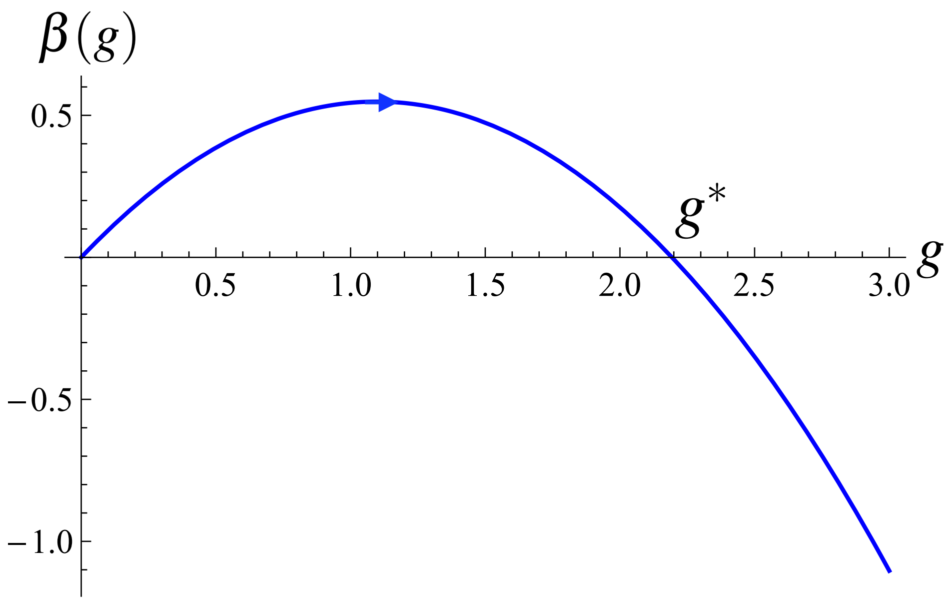
Perturbation theory for the effective scale-dependent parameters $g, m^2, \ldots$

β -function ($\epsilon = 4 - d$)

2-loop

$$\beta(g) := \partial_\ell g \equiv= -\frac{\mu \mathrm{d}}{\mathrm{d}\mu} g = \varepsilon g - a_1 g^2 + a_2 g^3 + \ldots$$

1-loop



ε -expansion

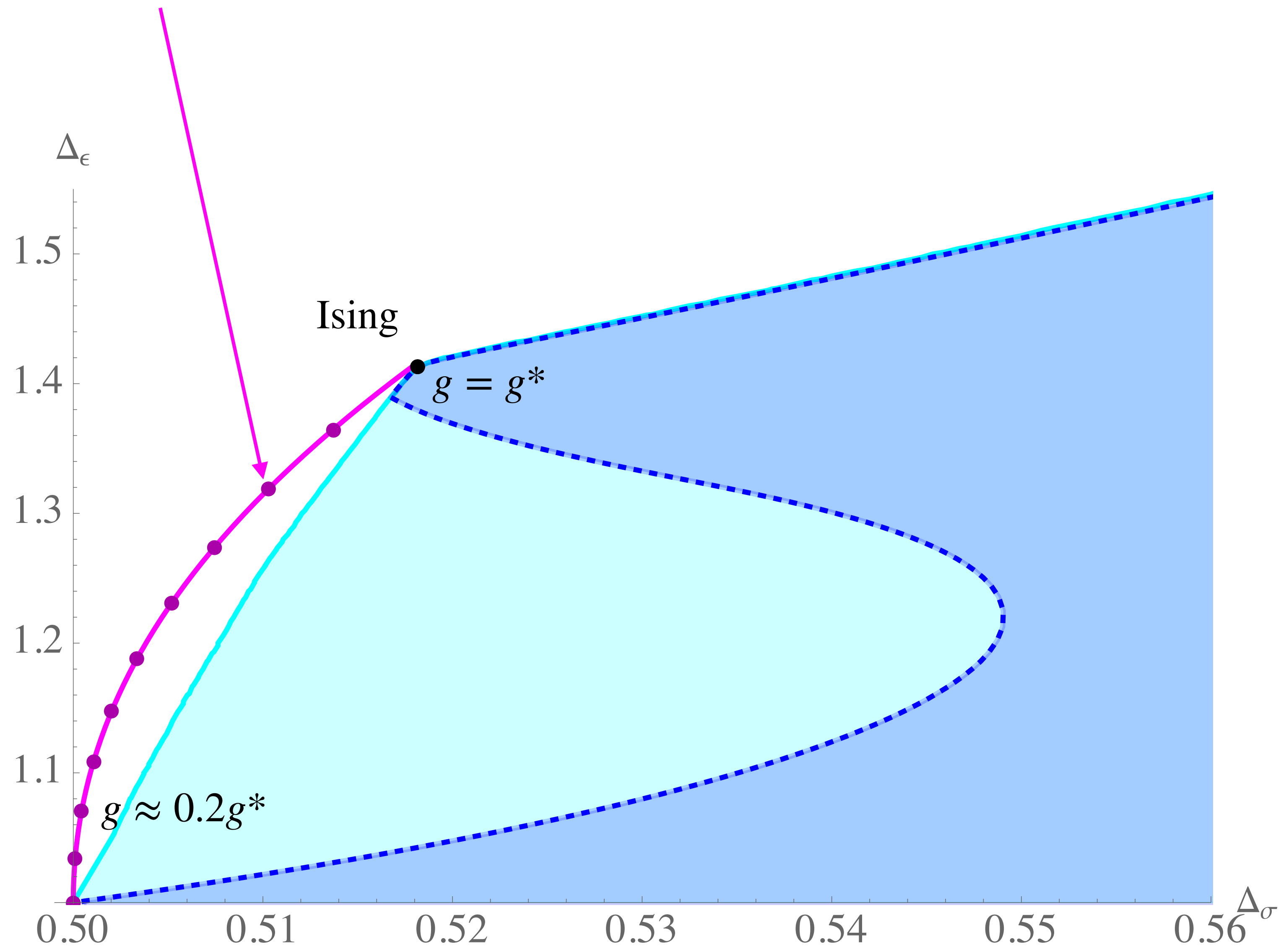
fixed point: $$g^* = \frac{\varepsilon}{a_1} + \frac{a_2}{a_1^3}\varepsilon^2 + \ldots \qquad \textbf{(1)}$$

Most relevant scalars ($n = 1$): $\sigma \equiv \phi$, $\varepsilon \equiv \phi^2$

$$\Delta_\sigma = \Delta_\sigma(g^*), \quad \Delta_\varepsilon = \Delta_\varepsilon(g^*)$$

(1) turns this into an ϵ-expansion

The RG trajectory in $d = 3$



- the theory on the trajectory is not conformally invariant.
- the trajectory moves through the forbidden region.

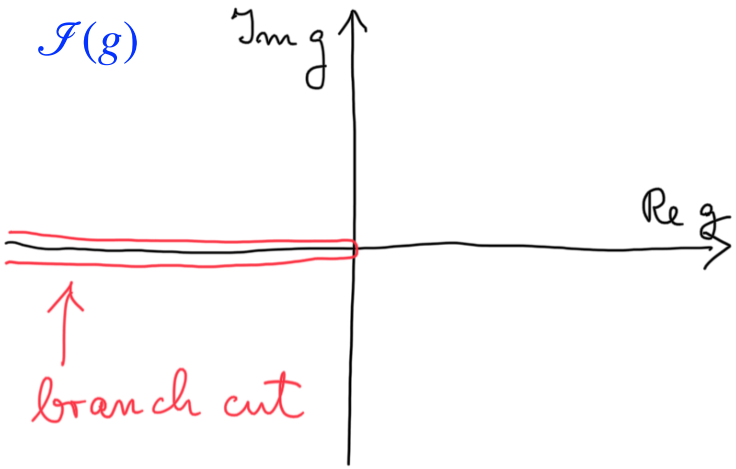
Asymptotic Series: A toy example

$$\mathcal{I}(g) := \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2 - gx^4} = \sum_{n=0}^{\infty} a_n (-g)^n, \qquad a_n = \frac{(4n)!}{2^{2n}\,(2n)!\,n!} \simeq \frac{2^{4n}}{\sqrt{2\pi n}} \times n!$$

Asymptotic behaviour can be obtained from saddle point

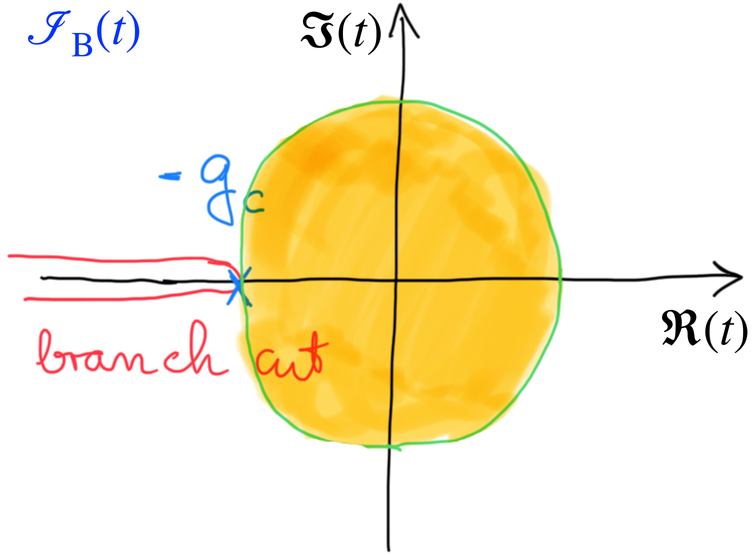
$$a_n = \frac{2}{n!} \int_0^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} (x^4)^n \,\mathrm{e}^{-x^2/2} = \frac{2}{n!} \int_0^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} \,\mathrm{e}^{-x^2/2 + 4n \ln x} = \frac{2\sqrt{n}}{n!} \int_0^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} \,\mathrm{e}^{-n[x^2/2 - 4\ln x] + 2n \ln(n)}$$

$(n!)^2$



Borel transform

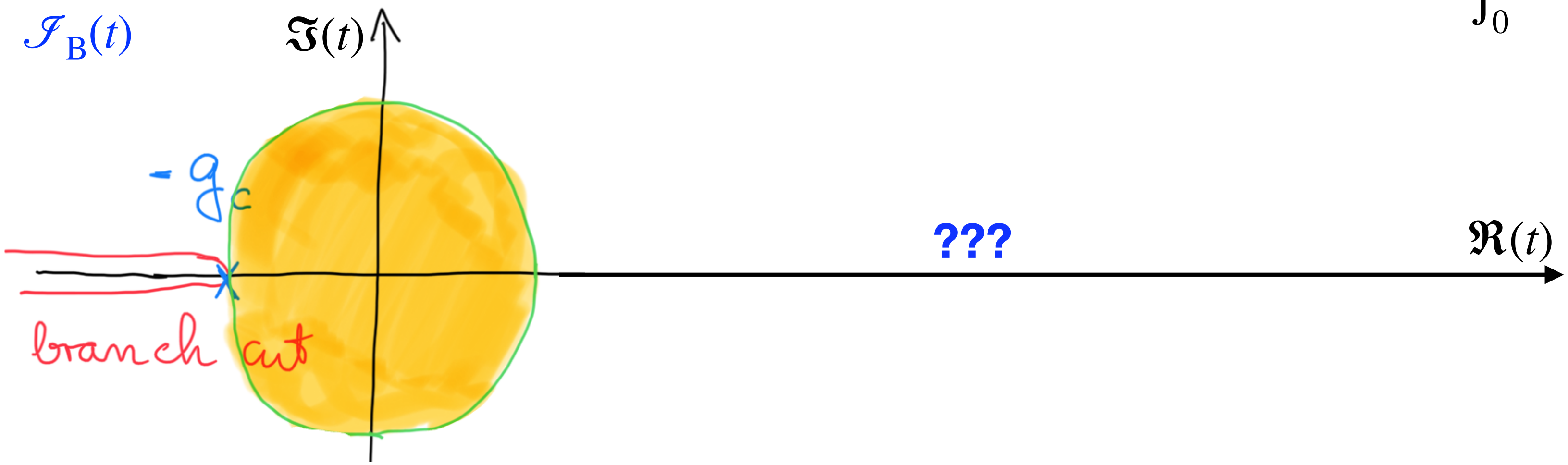
$$\mathcal{I}_{\mathrm{B}}(t) := \sum_{n=0}^{\infty} \frac{a_n (-t)^n}{n!}$$



Inverse Borel transform: $\mathcal{I}(g) = \int_0^{\infty} \mathrm{d}t \, \mathrm{e}^{-t} \mathcal{I}_{\mathrm{B}}(tg)$.

Field theory: works the same: saddle point is a function $\phi(x)$

Inverse Borel transform: $\mathscr{I}(g) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \mathscr{I}_{\mathrm{B}}(tg)$.



- Padé-resummation ("Padé-Borel"): unreliable due to spurious poles on the axis
- conformal transformation maps ∞ to g_c, s.t. integral is inside range of convergence. Standard method, works well.
  - estimate g_c self-consistently (Kompaniets-Wiese 2019) Phys. Rev. E 101 (2019) 012104, arXiv:1908.07502
  - Kompaniets-Panzer 2017: allow for several free parameters, and look for least sensitive point. Currently best method for error bars, computationally expensive. Phys. Rev. D 96 (2017) 036016, arXiv:1705.06483.
- Meijer-G resummation: fit $\mathscr{I}_{\mathrm{B}}(t)$ with hypergeometric function, yields a Meijer-G function for inverse Borel-transform. May have spurious poles on the axis. H. Mera, T. G. Pedersen and B.K. Nikolic, Phys. Rev. D 97 (2018) 105027.

Conclusion: If well done, resummation quality almost as for "normal" series.

An example from A. Aharony's talk:

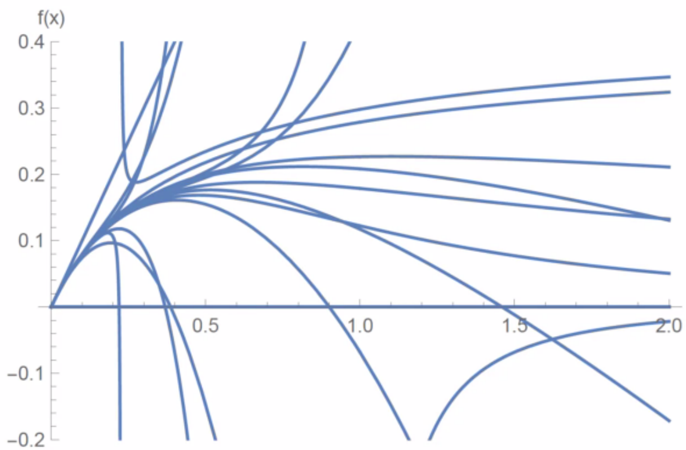
Zoom

$$4-\epsilon-n_c(\epsilon)=\epsilon-2.58848\epsilon^2+5.87431\epsilon^3-16.827\epsilon^4+56.62196\epsilon^5+\mathcal{O}[\epsilon^6].$$

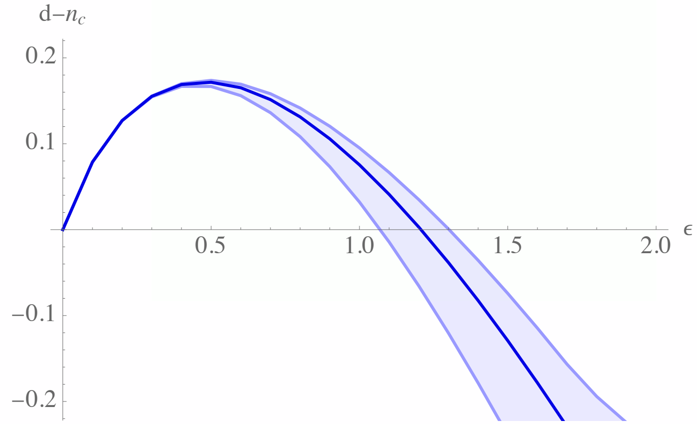
$$\begin{pmatrix} \dfrac{1-0.31906578597938484\text{`}\,\epsilon}{1+2.269409805252934\text{`}\,\epsilon} & \dfrac{1-4.551834202156295\text{`}\,\epsilon}{1-1.9633586109239758\text{`}\,\epsilon-10.956417728613014\text{`}\,\epsilon^2} & \dfrac{1+2.177564360014011\text{`}\,\epsilon}{1+4.76603995124633\text{`}\,\epsilon+6.462466193238685\text{`}\,\epsilon^2+5.557769889409281\text{`}\,\epsilon^3} \\ \dfrac{1+0.2760367726022195\text{`}\,\epsilon-1.5404084471683843\text{`}\,\epsilon^2}{1+2.8645123638345384\text{`}\,\epsilon} & \dfrac{1+2.684825951080864\text{`}\,\epsilon-2.308970659658309\text{`}\,\epsilon^2}{1+5.273301542313183\text{`}\,\epsilon+5.466529780426597\text{`}\,\epsilon^2} & 0 \\ \dfrac{1.\text{`}+0.7764628211312679\text{`}\,\epsilon-2.8357490590026666\text{`}\,\epsilon^2+2.9396586856390527\text{`}\,\epsilon^3}{1+3.364938412363587\text{`}\,\epsilon} & 0 & 0 \end{pmatrix}$$

(with Ora Entin-Wohlman)

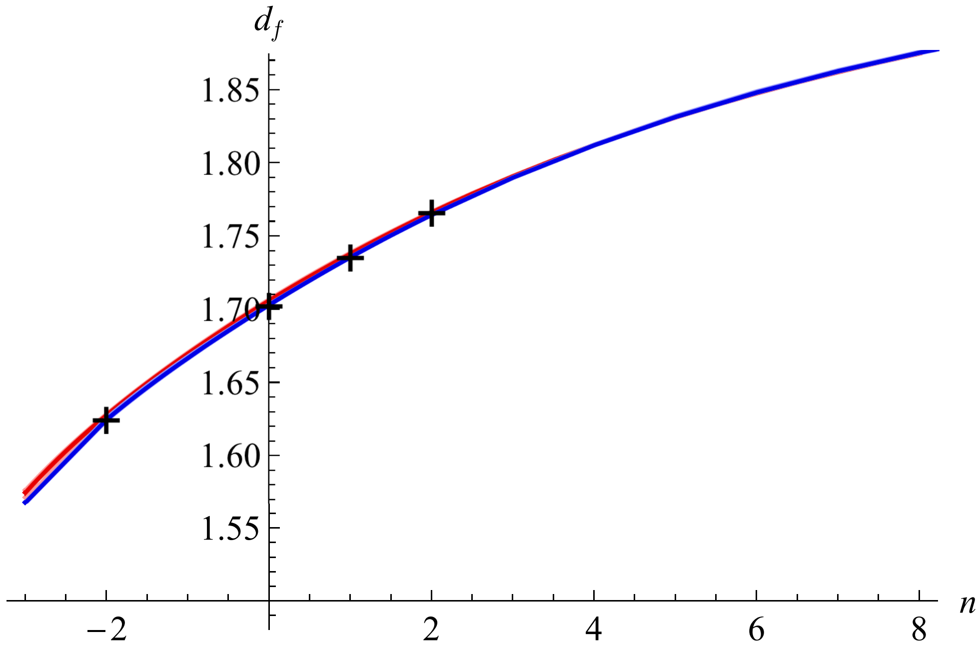
Padé



Borel-resummed

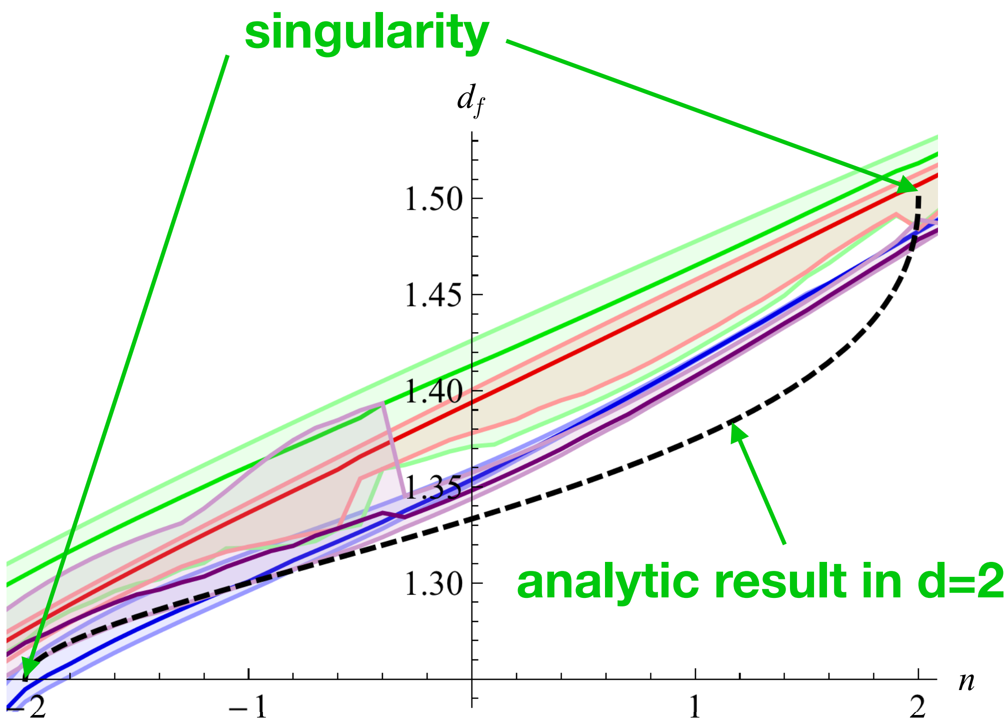


Resummation may be problematic due to analytic structure



d_f	n	SC	KP17	simulation
LERW	-2	1.6243(10)	1.623(6)	1.62400(5) [45]
SAW	0	1.7027(10)	1.7025(7)	1.701847(2) [24]
Ising	1	1.7353(10)	1.7352(6)	1.7349(65) [46]
XY	2	1.7644(10)	1.7642(3)	1.7655(20) [46, 47]

***d* = 3**



d_f	n	SC	KP17	CFT
LERW	-2	1.244(6)	1.188(55)	$5/4 = 1.25$
SAW	0	1.354(5)	1.350(8)	$4/3 \simeq 1.333$
Ising	1	1.416(1)	1.413(7)	$11/8 = 1.375$
XY	2	1.482(1)	1.480(4)	$3/2 = 1.5$

***d* = 2**

Redundant operators and rearrangement of states at the RG fixed point

the path integral $\mathscr{Z} = \prod_x \mathrm{d}\phi(x)\, e^{-\mathcal{S}[\phi]}$

is invariant under $\phi(x) \to \phi(x) + \delta\phi(x)$

the additional term in $\mathscr{Z}$ must vanish

$$\frac{\delta S[\phi]}{\delta\phi(x)} = -\nabla^2\phi(x) + g\phi^3(x) = 0$$

more general transformations $\phi(x) \to \phi(x) + f(\phi)\delta\phi(x)$

(Jacobian vanishes in dimensional regularisation, and cancels in expectations.)

$$f(\phi)\frac{\delta S[\phi]}{\delta\phi(x)} = f(\phi)\Big[-\nabla^2\phi(x) + g\phi^3(x)\Big] = 0$$

Redundant operators **(F. Wegner J. Physics C7 (1974) 2098)**

$\Longrightarrow$ rearrangement of states.

Models with long-ranged elasticity, and spectrum rearrangement

$$\mathcal{S}[\phi] = \int_x \frac{c_{\mathrm{LR}}}{2}\phi(x)(-\nabla^2)^{\frac{\alpha}{2}}\phi(x) + \frac{c_{\mathrm{SR}}}{2}[\nabla\phi(x)]^2 + \frac{m^2}{2}\phi(x)^2 + \frac{g}{4}\phi(x)^4$$

long-ranged elasticity $\Delta_\sigma = \dfrac{d-\alpha}{2}$

short-ranged elasticity **renormalises**

RG-flow

$$\partial_\ell c_{\mathrm{LR}} = 0$$

$$\partial_\ell c_{\mathrm{SR}} = \alpha - 2 + \eta(g)$$

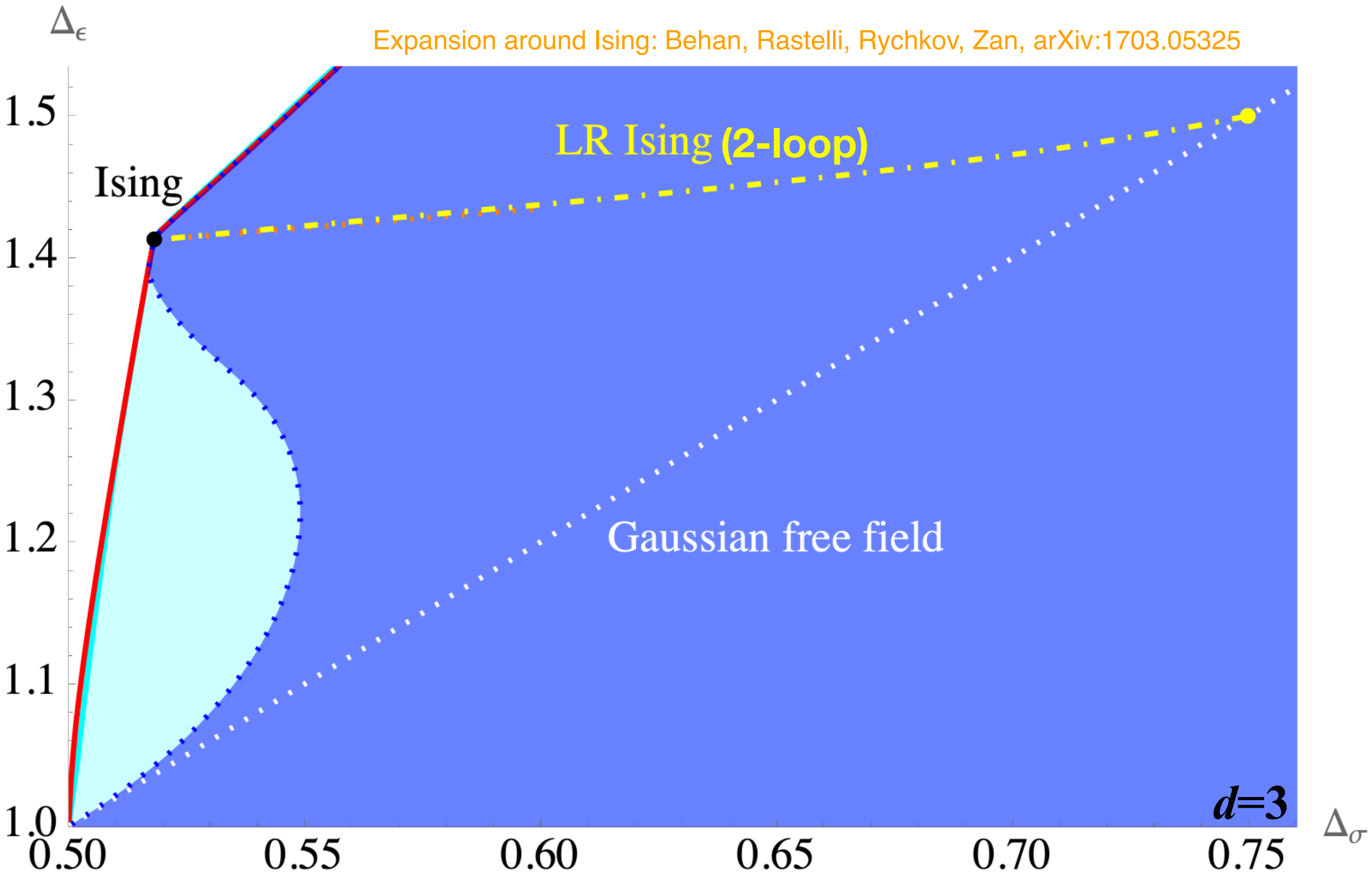
$\partial_\ell c_{\mathrm{SR}} < 0$ **LR**

$\partial_\ell c_{\mathrm{SR}} > 0$ **SR**

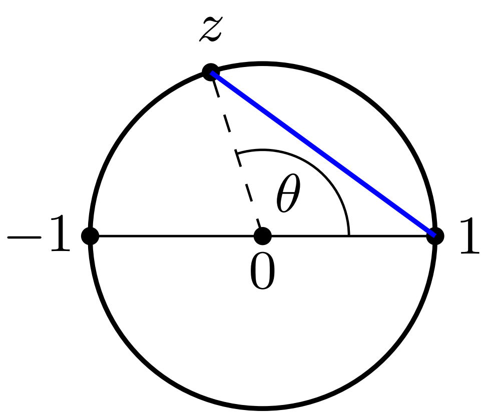
different redundant operator

$\Longrightarrow$

Spectrum Rearrangement upon reaching Ising



Experimental Realisation of LR elasticity



$$\mathcal{H}_{\text{boson}} = \frac{1}{8\pi}\int \mathrm{d}^2\vec{z}\,\left[\nabla\Phi(\vec{z})\right]^2 \ .$$

$$\langle\Phi(\vec{z})\Phi(\vec{z}')\rangle = -\ln|\vec{z}-\vec{z}'|^2$$

Restrict to the circle $|\vec{z}| = 1$

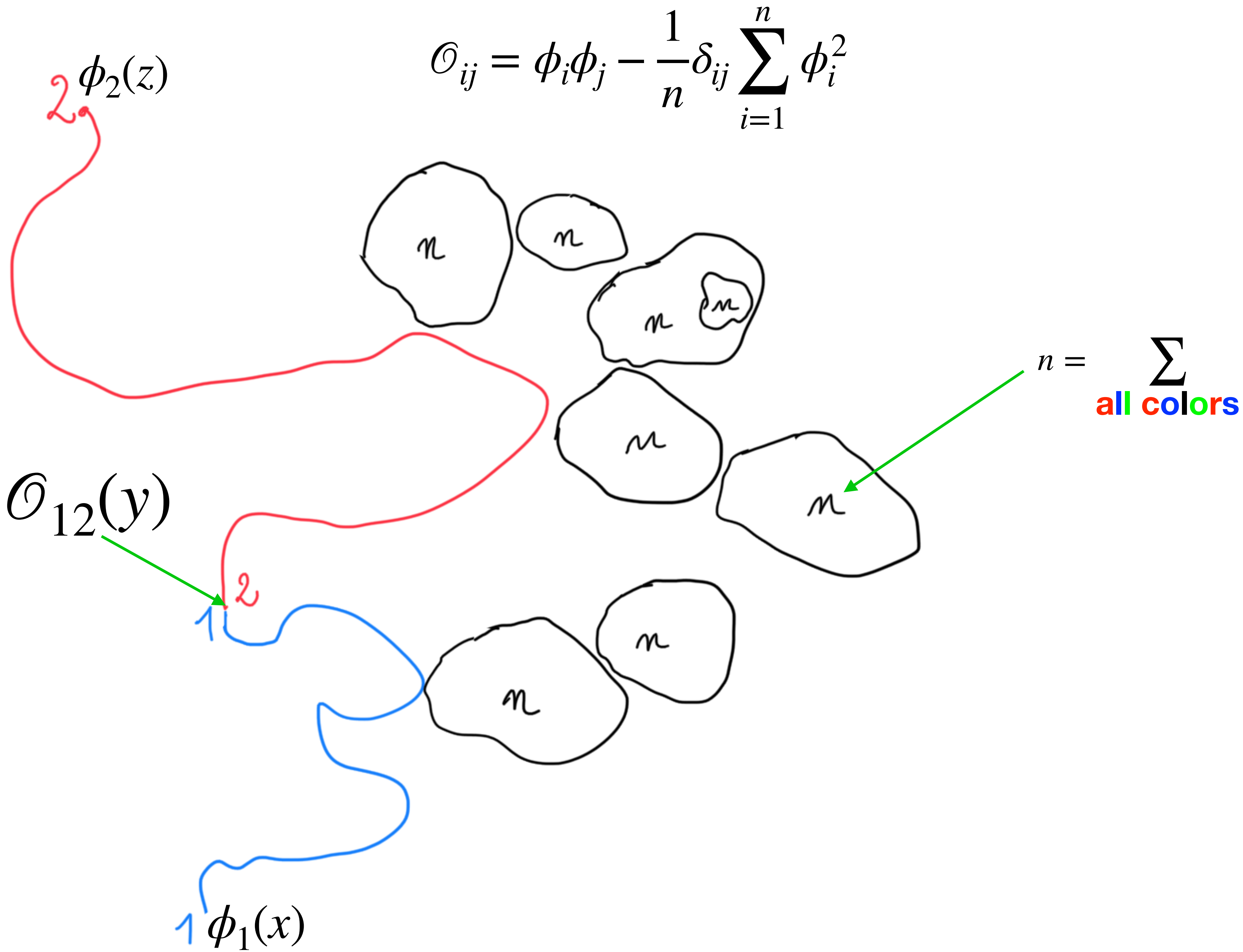
$$\phi(\theta) := \Phi(\mathrm{e}^{i\theta}) \implies \mathcal{O}(\theta) := \frac{1}{i}\phi'(\theta)$$

$$\langle\mathcal{O}(\theta)\mathcal{O}(\theta')\rangle = \frac{1}{2\sin^2(\frac{\theta-\theta'}{2})} \implies \mathcal{O}(\theta) = \text{ primary with } \Delta_{\mathcal{O}} = 1.$$

Warning: Gaussian free field (or its derivative) on the circle are not conformally invariant.

Curve-detecting operator for the $O(n)$ model

$$\mathcal{O}_{ij} = \phi_i \phi_j - \frac{1}{n}\delta_{ij}\sum_{i=1}^{n}\phi_i^2$$



Curve-detecting operator for the $O(n)$ model

$$\mathcal{O}_{ij} = \phi_i \phi_j - \frac{1}{n}\delta_{ij} \sum_{i=1}^{n} \phi_i^2$$

Applications

$n = -2$ loop erased random walks (LERW)

$n = 0$ self-avoiding polymers/walks (SAW)

$n = 1$ propagator line in the Ising model

$n = 2$ propagator line in the XY model

d_f	n	SC 6 loops	KP17	simulation
LERW	-2	$1.6243(10)$	$1.623(6)$	$1.62400(5)$ [45]
SAW	0	$1.7027(10)$	$1.7025(7)$	$1.701847(2)$ [24]
Ising	1	$1.7353(10)$	$1.7352(6)$	$1.7349(65)$ [46]
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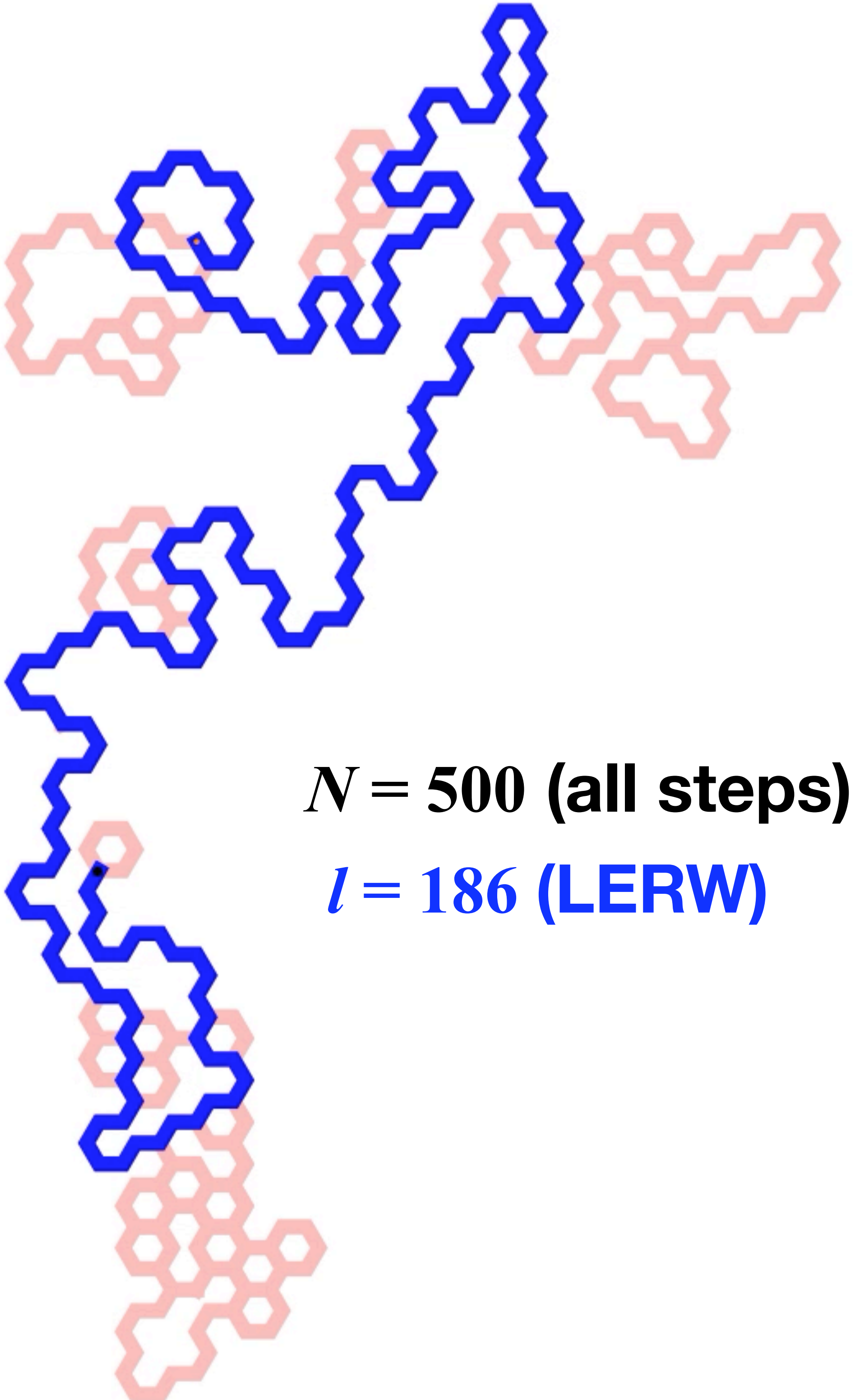


Loop-erased Random Walks (LERW)

- random walk
- erase a loop as soon as it is formed
- remains: loop-erased random walk (LERW)

R_g = radius of gyration

$R_g = N^{1/2} = l^{1/z}$

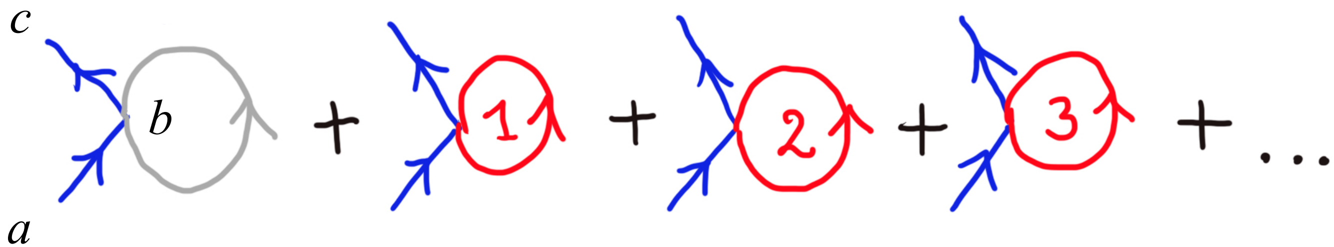
z = fractal dimension of LERW

$z = 5/4 \quad (d=2)$

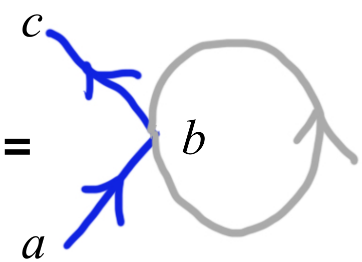
$z = 1.624\ldots \quad (d=3)$

Loop-erased random walks from field theory

$$\mathcal{P}(\gamma) = \sum_{\omega:\mathcal{L}(\omega)=\gamma} q(\omega) \;=$$



Multiply with $\mathcal{Z} = 1 -$  = **fermion partition function, gives**

$\mathcal{A}(\gamma)$ = 

= **fermion propagator (on the graph)**

Proven Theorem:

$$\mathcal{P}(\gamma) \times \mathcal{Z} \;=\; \mathcal{A}(\gamma) := q(\gamma) \sum_{L\in\mathcal{L}_\gamma} (-1)^{|L|} \prod_{C\in L} q(C)$$

Elements of Proof:

- **bubbles of non-intersecting loops factorise**
- **enlarge theory from 1 fermion to 2 fermions + 1 boson to detect path passing through**

From lattice action to field theory

A. Shapira and K.J. Wiese, SciPost Phys. 9 (2020) 063.

$$e^{-\mathcal{S}} = \prod_x \mathrm{e}^{-r_x \phi^*(x)\phi(x)} \Big[1 + \sum_y \beta_{xy} \phi^*(y)\phi(x)\Big] , \qquad \phi^*(y)\phi(x) := \sum_{i=1}^{3} \phi_i^*(y)\phi_i(x)$$

lattice action

ϕ_1, ϕ_2 = **fermions**

ϕ_3 = **boson**

$$\mathcal{S} = \sum_x \left[r_x \phi^*(x)\phi(x) - \ln\Big(1 + \sum_y \beta_{xy} \phi^*(y)\phi(x)\Big)\right]$$

leading term

$$\sum_x \Big[r_x \phi^*(x)\phi(x) - \sum_y \beta_{xy} \phi^*(y)\phi(x)\Big] = \sum_x \phi^*(x)[m_x^2 - \nabla_\beta^2]\phi(x)$$

$$m_x^2 = r_x - \sum_y \beta_{yx}, \qquad \nabla_\beta^2 \phi(x) = \sum_y \beta_{yx}[\phi(y) - \phi(x)].$$

subleading term

$$\frac{1}{2} \sum_x \Big[\sum_y \beta_{xy} \phi^*(y)\phi(x)\Big]^2 = \frac{g}{2} \sum_x \Big[\phi^*(x)\phi(x)\Big]^2 + \ldots , \qquad g := \Big[\sum_y \beta_{xy}\Big]^2$$

= action of ϕ^4-theory: 2 fermions and 1 boson, or -1 complex boson OR -2 real bosons

O(-2)

“almost” free theory $$\Delta_\sigma = \frac{d-2}{2}, \; \Delta_\epsilon = d-2 \qquad \text{(1)}$$

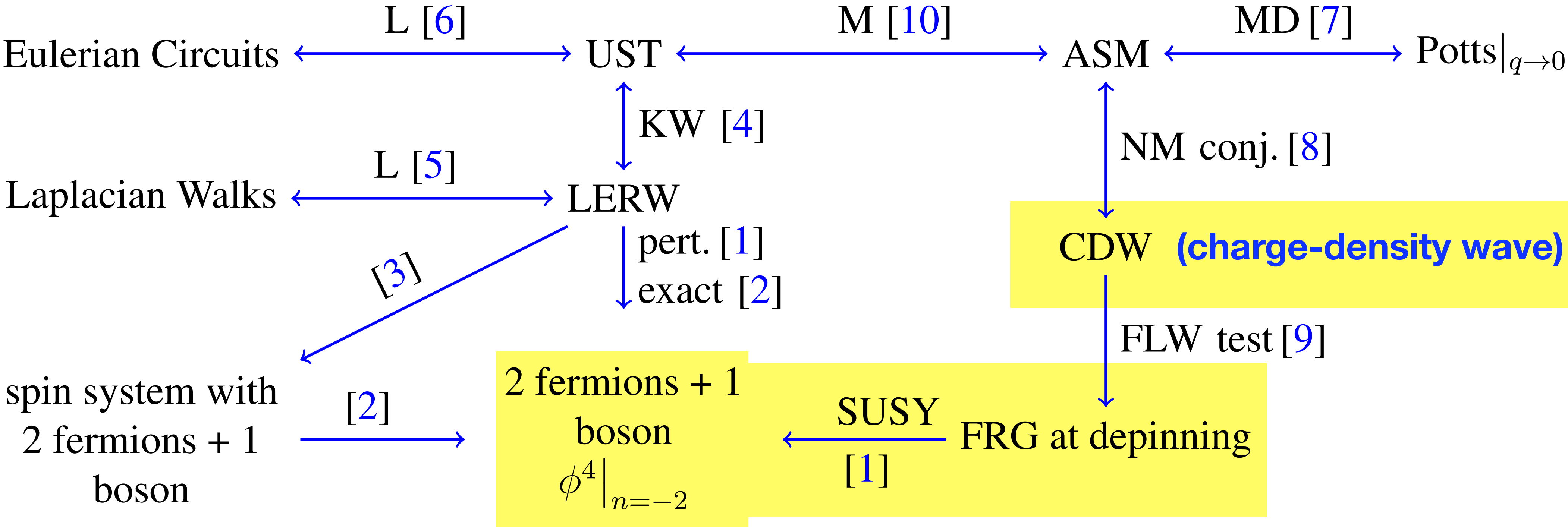
$$\Delta_{\mathcal{O}_{ij}} = 1.37600(5), \qquad \mathcal{O}_{ij} = \phi_i \phi_j - \frac{1}{n}\delta_{ij}\sum_{i=1}^{n}\phi_i^2$$

$$\Delta_{\epsilon'} = 3.82(1)$$

one family of fermions is free → (1)

does not talk about LERW observable: need 2 complex fermions + 1 complex boson

Relations



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Field Theory for Charge Density Waves (CDW)

- semi-conductor devices may have an instability for a periodic modulation of the charge density $\longrightarrow$ CDW

$$\mathscr{H}[u] := \int_x \frac{1}{2}[\nabla u(x)]^2 + \frac{m^2}{2}[u(x) - w]^2 + V\big(x, (u(x)\big)$$

disorder

- disorder force correlator

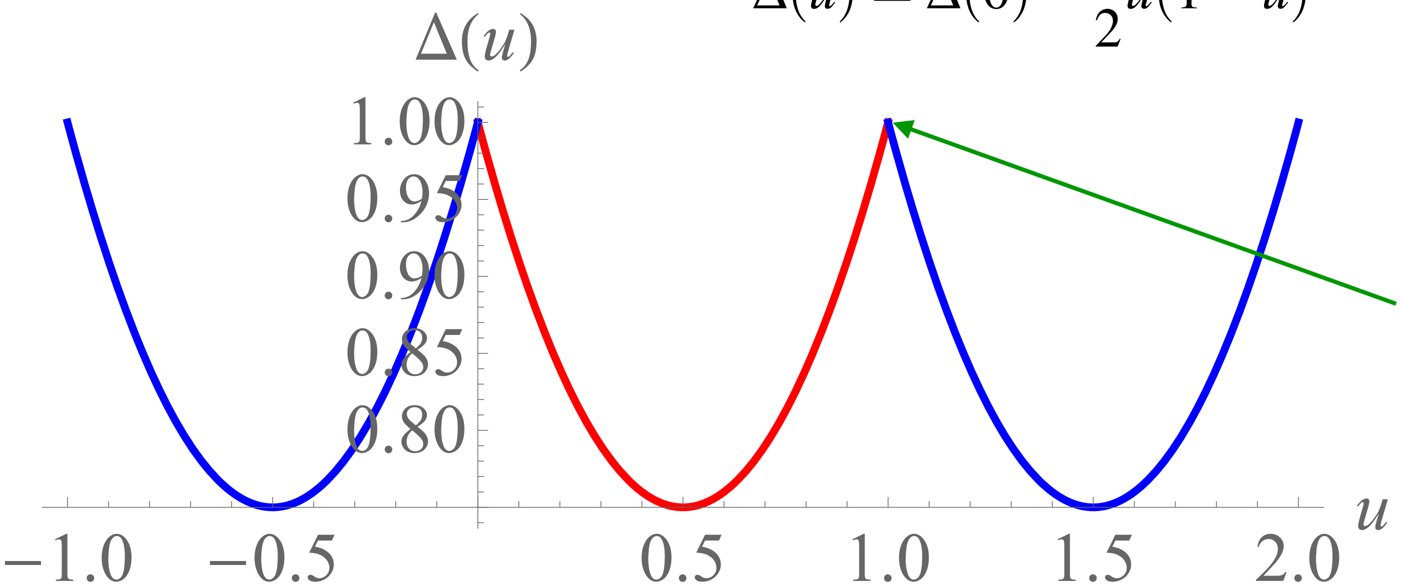
$$\overline{\partial_u V(x,u)\partial_{u'} V(x',u')} = \delta^d(x-x')\Delta(u-u')$$

- renormalizes under RG

$$-\frac{m\mathrm{d}}{\mathrm{d}m}\Delta(u) = (\varepsilon - 2\zeta)\Delta(u) + \zeta u \Delta'(u) - \partial_u^2\left[\frac{1}{2}\Delta(u)^2 - \Delta(u)\Delta(0)\right]$$

CDW: $\zeta = 0$ and periodic fixed point $\Delta(u)$, which is piecewise

$$\Delta(u) = \Delta(0) - \frac{g}{2}u(1-u) \qquad (g = \mathcal{O}(\epsilon))$$



$$|\Delta'(0^+)| = m^4 L^{-d} \frac{\langle S^2 \rangle}{2\langle S \rangle}$$

avalanche moments

Charge Density Waves (CDW) → ϕ^4-theory at *N*=-1

Action at depinning

$$\mathcal{S}^{\rm CDW} = \int_{x,t} \tilde u(x,t)(\partial_t - \nabla^2 + m^2)u(x,t) - \frac{1}{2}\int_{x,t,t'} \tilde u(x,t)\tilde u(x,t')\Delta\big(u(x,t)-u(x,t')\big).$$

FRG fixed point function for CDWs at depinning

$$\Delta(u) = \Delta(0) - \frac{g}{2}u(1-u)$$

difference $\phi(x)$ between 2 copies

Keep only leading term ~ $g\, u^2/2$

$$\mathcal{S}^{\rm CDW}_{\rm simp} := \int_{x,t} \tilde u(x,t)(\partial_t - \nabla^2 + m^2)u(x,t) - \frac{g}{4}\int_{x,t,t'} \tilde u(x,t)\tilde u(x,t')\big[u(x,t)-u(x,t')\big]^2$$

Redo with Supersymmetry

$$\mathcal{S} = \int_x \tilde\phi(x)(-\nabla^2+m^2)\phi(x) + \tilde u(x)(-\nabla^2+m^2)u(x) + \sum_{a=1}^{2}\bar\psi_a(x)(-\nabla^2+m^2)\psi_a(x)$$

$$+\frac{g}{2}\tilde u(x)\phi(x)\Big[\bar\psi_2(x)\psi_2(x) - \bar\psi_1(x)\psi_1(x)\Big] - \frac{g}{8}\tilde u(x)^2\phi(x)^2$$

decouple

$$+\frac{g}{2}\left[\tilde\phi(x)\phi(x) + \bar\psi_1(x)\psi_1(x) + \bar\psi_2(x)\psi_2(x)\right]^2 .$$

multiplet out of 1 boson + 2 fermions = -1 boson

Conclusions

O(n) model at n = -2
= loop-erased random walks
= CDWs at depinning
= Abelian sandpiles
… more interesting physics hiding there …