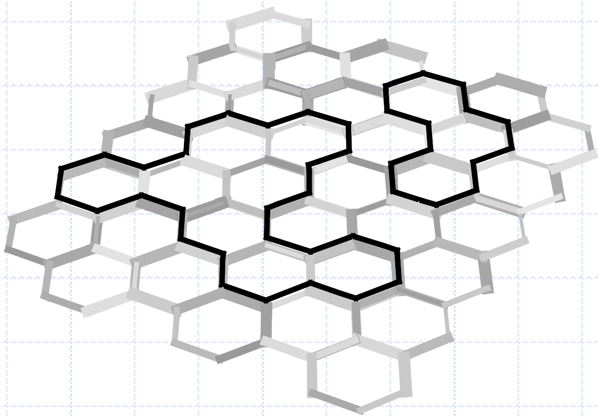
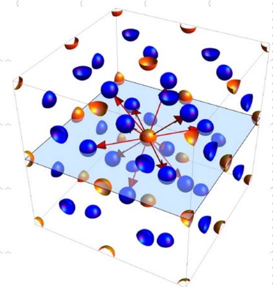
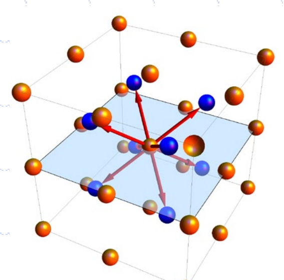
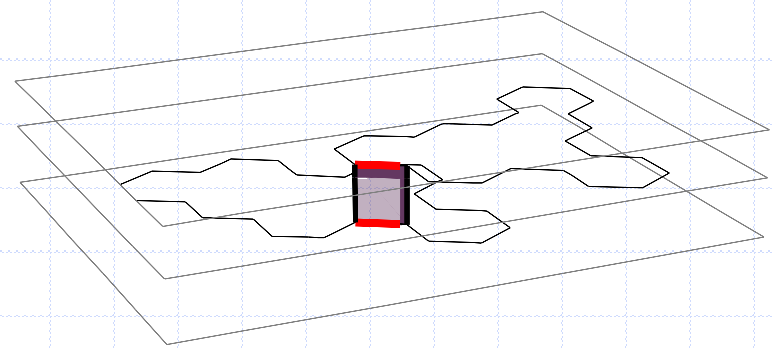
RG flows in Replica Coupled CFTs

$$\mathcal{H}_{ij} = -\left[K_1\left(\delta_{\sigma_i\sigma_j} + \delta_{\tau_i\tau_j} + \delta_{\eta_i\eta_j}\right) + K_2\left(\delta_{\sigma_i\sigma_j}\delta_{\tau_i\tau_j} + \delta_{\tau_i\tau_j}\delta_{\eta_i\eta_j} + \delta_{\sigma_i\sigma_j}\delta_{\eta,\eta'}\right) + K_3\delta_{\sigma_i\sigma_j}\delta_{\tau_i\tau_j}\delta_{\eta_i\eta_j}\right]$$





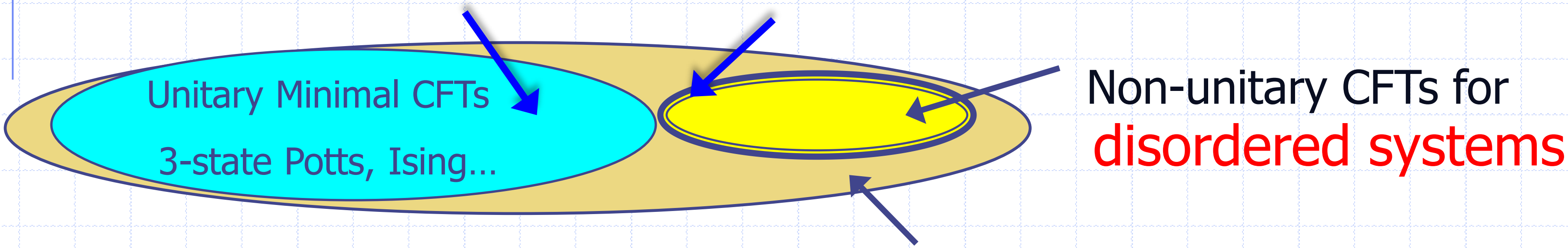




Hirohiko Shimada (NIT, Tsuyama)

Introduction: Replica CFTs in 2d

- Using the conformal invariance, the RG fixed points for many two-dimensional critical systems are now well-understood (BPZ 1984).
- However, the fixed points for disordered systems, even in two-dimensions, are poorly-understood.



Other non-unitary CFTs (q-state Potts and O(n), $\mathbb{Z}_k$-Parafermions…for generic q, n, k)

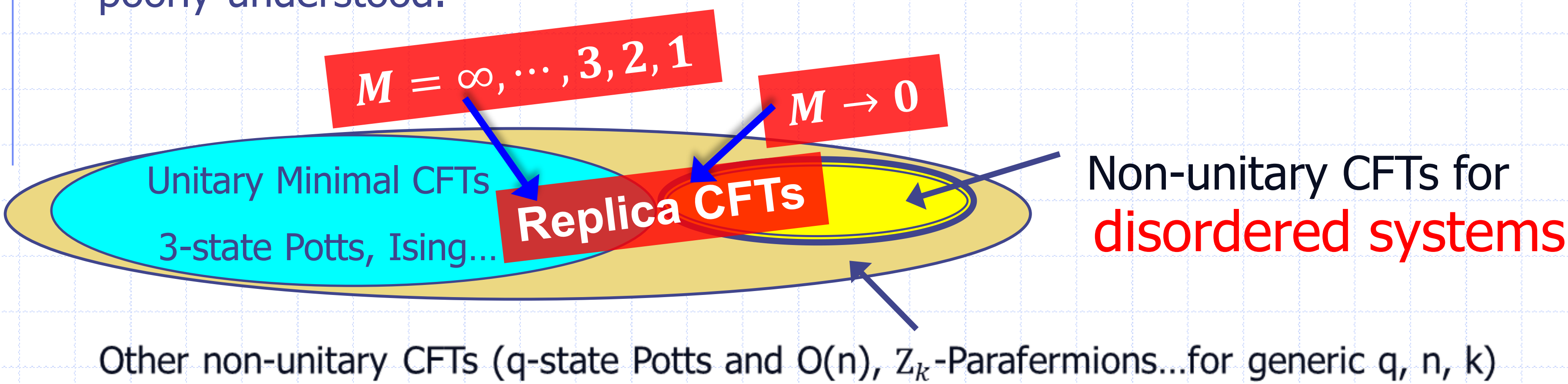
Lack of the unitarity ==> Needs deeper understanding of "symmetry".
Effective symmetries of disordered systems: the Replica or SUSY(supergroup).

$$\overline{\log Z} = \lim_{M \to 0} \overline{(Z^M - 1)/M}$$

M : number of replicas

Introduction: Replica CFTs in 2d

- Using the conformal invariance, the RG fixed points for many two-dimensional critical systems are now well-understood (BPZ 1984).
- However, the fixed points for disordered systems, even in two-dimensions, are poorly-understood.



Lack of the unitarity ==> Needs deeper understanding of "symmetry".
Effective symmetries of disordered systems: the Replica or SUSY(supergroup).

Outstanding unsolved problems:

- Disordered CFTs @ $M \to 0$ (Non-unitary) -------interesting though difficult in many ways
- CFTs for M-coupled q-state Potts models (Unitary) with $M = 3,4,5,\cdots$
  $q = 3,4$

e.g., 3-state Potts model CFT with an S_4-extended symmetry $\in$ Bootstrap targets?

Epsilon expansion from Ising CFT (d=2 fixed) varying the size of internal symmetry

In 2d, the Ising model saturates the Harris criterion.
The random-bond Ising model (RBIM) plays a pivotal role.

$$\sum_{a\neq b}^{M} \mathcal{E}^a(x)\mathcal{E}^b(x)$$ $2\Delta_\varepsilon = 2$ (marginal)

Randomness is relevant if $\epsilon \propto 2 - 2\Delta_\varepsilon > 0$

Ising model = "O($n = 1$) model" = "$q = 2$-state Potts model"

$(q-2)$–expansion

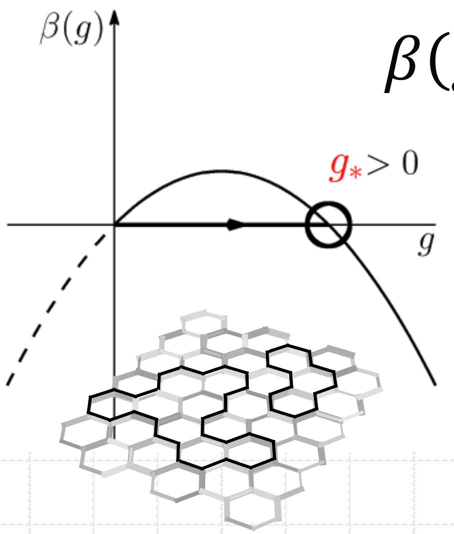
A.W.W. Ludwig, Nucl. Phys. B285 (1987) 97.

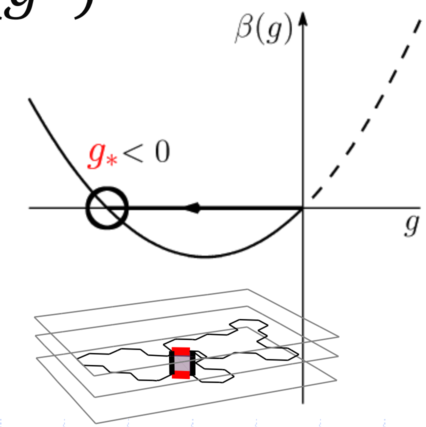
Small deviation from the Ising model: $\epsilon \propto (1-n)$ or $(q-2)$

RG beta function $\beta(g) = \epsilon g + [(M-2) + C^2(\epsilon)]g^2 + O(g^3)$ (schematic)

$C(\epsilon) = C^{\varepsilon}_{\varepsilon\varepsilon}(\epsilon)$ vanishes for the 2d Potts critical points (self-duality)

$$\beta(g) = \epsilon g + (M-2)g^2 + O(g^3)$$





IR fixed point is located at

$$g_* = \frac{\epsilon}{2-M} + O(\epsilon^2)$$

M-coupled q-state Potts models (Unitary) with $M = 3,4,5,\cdots$

Disordered CFTs @ $M \to 0$ (Non-unitary)

$M = 2, q > 2$ is integrable and massive (Vaysburd 95)

Loop and spin models belong to the same universality class

O(n) loop model …Polymer(n=0), Ising(n=1), XY(n=2) as special cases
Belongs to the same universality class as a spin model in which each site has n -component spin $s_i = (s_i^{(1)}, \cdots, s_i^{(n)})$ interacting with the nearest-neighbor spins:

$$Z(x,n) = \mathrm{Tr}_{s_i} \prod_{\langle i,j \rangle} (1 + x s_i \cdot s_j)$$

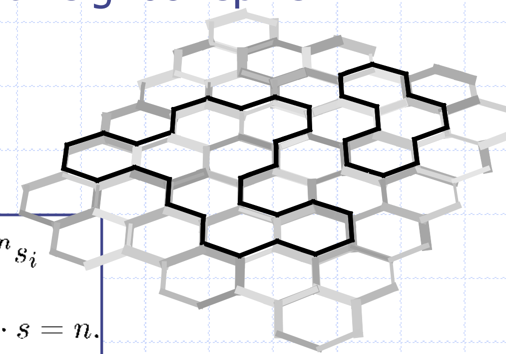
$$= \int \prod_i \mu(s_i) d^n s_i \prod_{\langle i,j \rangle} (1 + x s_i \cdot s_j)$$

$$= \sum_{\text{loops}} x^{\#\text{bonds}} n^{\#\text{loops}}.$$

$$\mathrm{Tr}_{s_i} \equiv \int \prod_i \mu(s_i) d^n s_i$$

$$\mathrm{Tr}_s\ 1 = 1, \qquad \mathrm{Tr}_s\ s \cdot s = n.$$

$$\mathrm{Tr}_s\ s = 0,$$

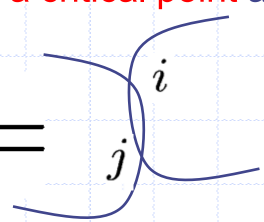


--becomes particularly simple on the honeycomb lattice.

- Each closed loop (~particle trajectory) has a weight n, (which makes sense for $n \in \mathbb{R}$) whereas each bond (a segment of loops) has a weight x.

- When $|n| \leq 2$, the model has a critical point at $x_c = \sqrt{2 + \sqrt{2-n}}$, where

$$(s_i \cdot s_j)^2 =$$



is irrelevant in RG

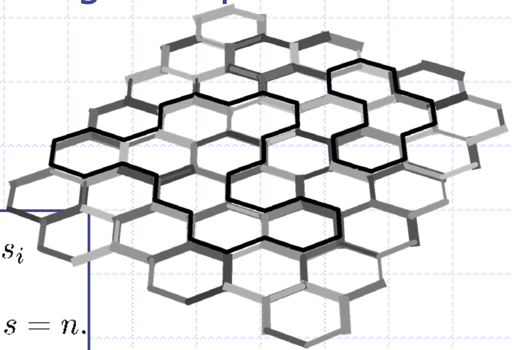
Disordered O(n) Loop Model on a Lattice

O(n) loop model …Polymer(n=0), Ising(n=1), XY(n=2) as special cases

Belongs to the same universality class as a spin model in which each site has n-component spin $s_i = (s_i^{(1)}, \cdots, s_i^{(n)})$ interacting with the nearest-neighbor spins:

$$Z(x,n) = \mathrm{Tr}_{s_i} \prod_{\langle i,j\rangle} (1 + x s_i \cdot s_j)$$
$$= \int \prod_i \mu(s_i) d^n s_i \prod_{\langle i,j\rangle} (1 + x s_i \cdot s_j)$$
$$= \sum_{\text{loops}} x^{\#\text{bonds}} n^{\#\text{loops}}.$$

$$\mathrm{Tr}_{s_i} \equiv \int \prod_i \mu(s_i) d^n s_i$$
$\mathrm{Tr}_s\ 1 = 1,\quad \mathrm{Tr}_s\ s \cdot s = n.$
$\mathrm{Tr}_s\ s = 0,$



--becomes particularly simple on the honeycomb lattice.

- Each closed loop (~particle trajectory) has a weight n, (which makes sense for $n \in \mathbb{R}$) whereas each bond (a segment of loops) has a weight x.

- Models with quenched disorder: $Z[\{x\},n] = \mathrm{Tr}_{s_i} \prod_{\langle i,j\rangle} (1 + x_{ij} s_i \cdot s_j)$

x_{ij} different from link to link; independently respects some distribution function.

e.g. $P(x_{ij}) = [p\delta(x_{ij} - x_1) + (1-p)\delta(x_{ij} - x_2)]\,.$

Strong and weak bonds

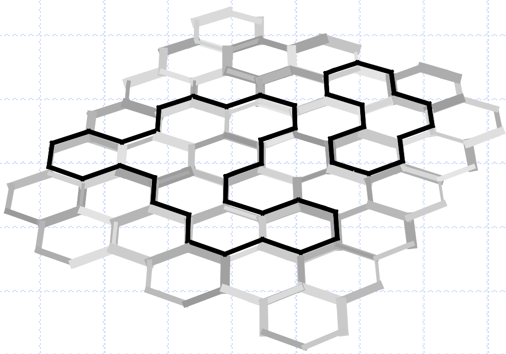
Disordered O(n) Model Formulated on a Lattice

- Models with quenched disorder: $Z[\{x\},n] = \mathrm{Tr}_{s_i} \prod_{\langle i,j \rangle} (1 + x_{ij} s_i \cdot s_j)$

x_{ij} different from link to link; independently respects some distribution function.

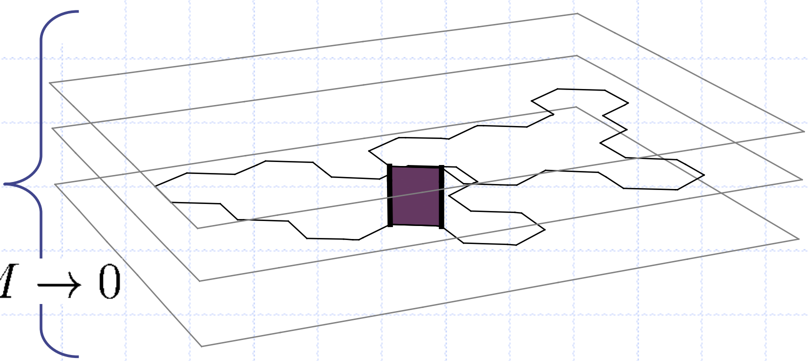
e.g. $P(x_{ij}) = [p\delta(x_{ij} - x_1) + (1-p)\delta(x_{ij} - x_2)]\,.$

Weak and strong bonds



Replica trick

$$\overline{\ln Z} = \lim_{M \to 0} \frac{\overline{Z^M} - 1}{M}.$$



$M \to 0$

Coupled vertically but horizontally homogeneous

Approaching the Continuum Limit of the Disordered O(n) model

- Without disorder, the following relations summarize the vicinity of the critical point (=dilute phase):

$$\prod_{\langle i,j\rangle}(1+ts_i\cdot s_j)\sim \exp\left[\beta\sum_{\langle i,j\rangle}s_i\cdot s_j\right]\to \exp\left[-S_{\rm CFT}+m\int d^2x\ \mathcal{E}(x)\right].\qquad m=(T-T_c)/T_c$$

bond $s_i\cdot s_j\to\mathcal{E}(x)$ energy operator; spin $s_i\to\sigma(x)$ spin operator

- Disordered models

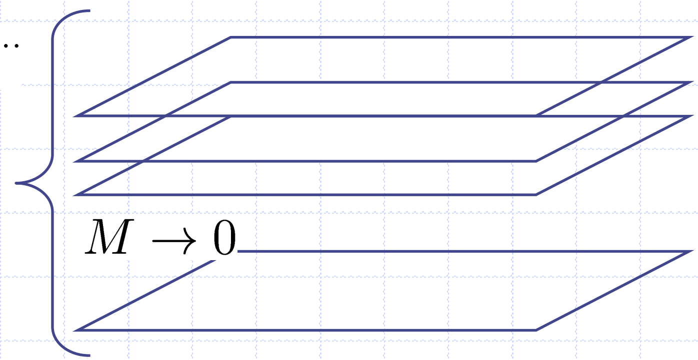
$$Z=\mathrm{Tr}_{s_i}\ e^{-H_0-\int m(x)\mathcal{E}(x)d^2x}.$$

The bond-strength distribution is assumed to be short-ranged and approximately Gaussian:

$$\overline{m(x)m(y)}=g_0^2\ \delta(x-y);\quad P(m(x))\ =\ e^{-\frac{1}{2g_0}(m(x)-m_0)^2+\cdots}$$

$$Z^M=\mathrm{Tr}_{s_i^{(a)}}\ \exp\left[-\sum_{a=1}^M H_0^{(a)}-\int m(z)\sum_{a=1}^M\mathcal{E}^a(x)d^2x\right]$$

$$\overline{Z^M}=\int\prod_x dm(x)P(m(x))\ Z^M.$$



- We have the following effective Hamiltonian; note that this contains composite operators.

$$\overline{Z^M}=\mathrm{Tr}_{s_i^{(a)}}\ e^{-\mathcal{H}_{\it eff}},$$

$$\mathcal{H}_{\it eff}=\sum_{a=1}^M H_0^{(a)}+\int d^2x\left[m_0\sum_{a=1}^M\mathcal{E}^a(x)-g_0\sum_{a,b=1}^M\mathcal{E}^a(x)\mathcal{E}^b(x)\quad-\sum_{k=3}^\infty\frac{\xi_k}{k!}\left(\sum_{a=1}^M\mathcal{E}^a(x)\right)^k\right]$$

Energy/spin operator are identified with certain Kac primary fields; Correlation fns. of these are expressed in terms of vertex operators.

- The identification of continuum fields (Dotsenko-Fateev84, Batchelor89):

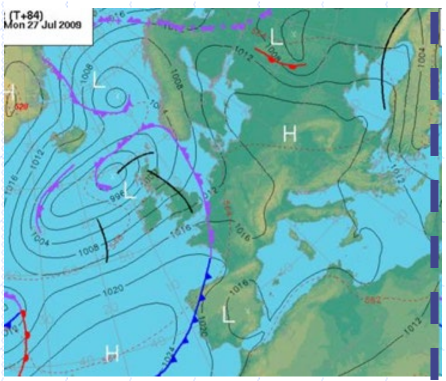
$$\begin{cases} s_i \cdot s_j \to \mathcal{E}(x) \longrightarrow V_{\alpha_{1,3}}(x) \\ s_i \longrightarrow \sigma(x) \longrightarrow V_{\alpha_{p-1,p}}(x) \end{cases} \quad \left(p = \frac{1/2}{1-(\alpha_-(n))^2} \right)$$

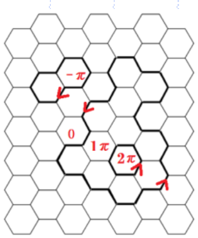
$$\alpha_{1/2,0} = \alpha_{p-1,p}$$

vertex operator operators

$$V_\alpha(x) \equiv e^{i\alpha\varphi(x)}$$

The charge for the Kac primary $\phi_{r,s}$ is $\alpha_{r,s} \equiv \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_-$, where $\alpha_\pm = \alpha_\pm(n)$ are determined s.t. $\alpha_+\alpha_- = -1$ and $n = -2\cos(\pi/\alpha_-^2)$





Lattice O(n) model → Continuum (CFT)

(Oriented-)Loop rep. → Coulomb gas

Discrete height fn. $H(z) = 0, \pm\pi, \pm 2\pi \cdots$

Coarse graining the height fn.

→ Gaussian free field $\varphi(x)$

Continuum Limit of the Disordered O(n) model

$$Z^M = \mathrm{Tr}_{s_i^{(a)}} \exp\left[-\sum_{a=1}^{M} H_{\mathrm{CFT}}^{(a)} - \int m(x) \sum_{a=1}^{M} \mathcal{E}^a(x) d^2x\right]$$

Bond $s_i \cdot s_j \to \mathcal{E}(x)$ energy operator

$$\overline{Z^M} = \int \prod_x dm(x) P(m(x))\ Z^M.$$

$$\overline{m(x)m(y)} = g_0^2\ \delta(x-y);$$

- By taking advantage of the identification $\mathcal{E} \to \phi_{1,3}$ primary field in the O(n) CFT, and

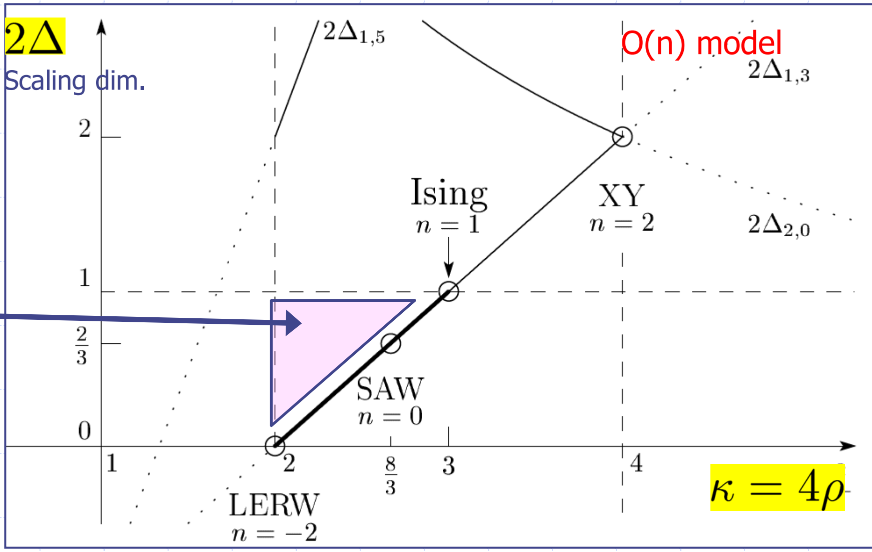
the fusion rule: $\phi_{1,3} \cdot \phi_{1,3} \sim \phi_{1,1} + \phi_{1,3} + \phi_{1,5}$

$\mathcal{E} \cdot \mathcal{E} \sim I + \mathcal{E} + \mathcal{E}'$

$$\sum_{k=3}^{\infty} \frac{\xi_k}{k!} \left(\sum_{a=1}^{M} \mathcal{E}^a(x)\right)^k \longrightarrow \text{Irrelevant}$$

$$\hat{g}_0 \sum_{a,b=1}^{M} \mathcal{E}^a(x)\mathcal{E}^b(x) \longrightarrow \text{Slightly relevant}$$

$$\hat{m}_0 \sum_{a=1}^{M} \mathcal{E}^a(x) \longrightarrow \text{Relevant}$$



$n = -2\cos(\pi/\rho)$

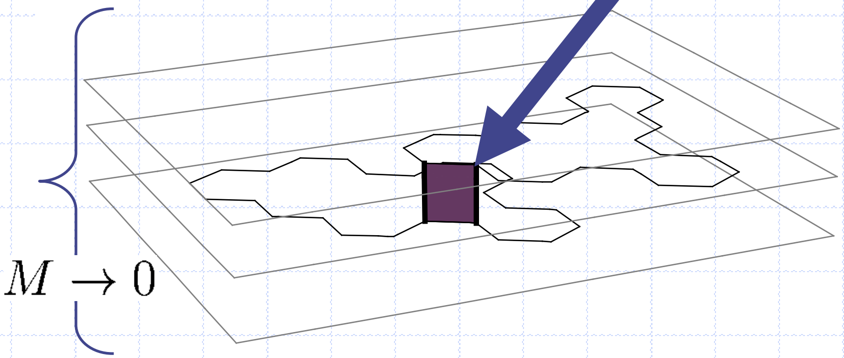
one obtains the following effective Hamiltonian:

Conformal perturbation theory
Epsilon-expansion around the Ising model (n=1)

$$\overline{Z^M} = \mathrm{Tr}_{s_i^{(a)}} \exp\left[-\sum_{a=1}^{M} H_0^a - m_0 \int \sum_{a=1}^{M} \mathcal{E}^a(x) d^2x + g_0 \int \sum_{a\neq b}^{M} \mathcal{E}^a(x)\mathcal{E}^b(x) d^2x\right]$$

$$\mathcal{H}_0 = \sum_{a=1}^{M} H_0^a, \qquad \mathcal{H}_{int} = \int \sum_{a\neq b}^{M} \mathcal{E}^a(x)\mathcal{E}^b(x) d^2x.$$

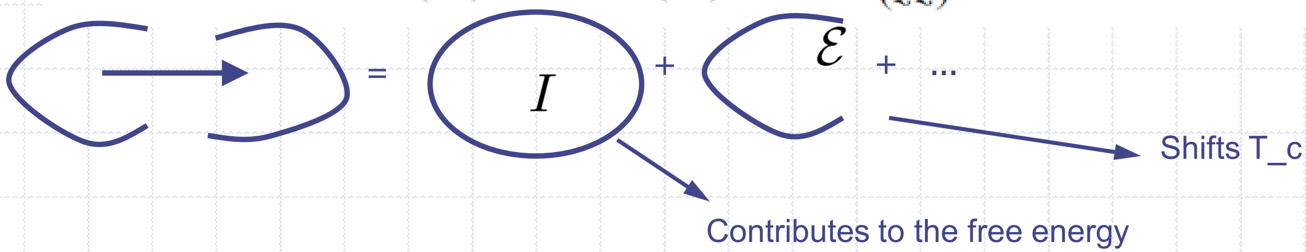
Interlayer-coupling btw replicas



Coupled vertically but horizontally homogeneous

Interpretation of the first two terms in the OPE

$$\mathcal{E}(0,0)\cdot\mathcal{E}(z,\bar{z}) = \frac{C^{I}_{\mathcal{E}\mathcal{E}}}{(z\bar{z})^{2\Delta_{\mathcal{E}}}} I + \frac{C^{\mathcal{E}}_{\mathcal{E}\mathcal{E}}}{(z\bar{z})^{\Delta_{\mathcal{E}}}}\mathcal{E} + \frac{C^{\mathcal{E}'}_{\mathcal{E}\mathcal{E}}}{(z\bar{z})^{2\Delta_{\mathcal{E}}-\Delta'_{\mathcal{E}}}}\mathcal{E}' + \cdots,$$



$$\left(C^{\mathcal{E}}_{\mathcal{E}\mathcal{E}}(\rho)\right)^2 = \left[2(1-2\rho)^2 \frac{\gamma^{\frac{3}{2}}(\rho)}{\gamma^2(2\rho)}\frac{\gamma^{\frac{1}{2}}(2-3\rho)}{\gamma(3-4\rho)}\right]^2,$$

$$\sim 12^2 \cdot 0.01305 \cdot \epsilon^2.$$

has a pole at $\epsilon = 1/12$ $(n=0)$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$$

- The OPE structure constant encodes selection rules:

(1) $C^{\mathcal{E}}_{\mathcal{E}\mathcal{E}} = 0$ at $n = 1$; the Ising model is invariant under the Kramars-Wannier duality.

(2) $C^{\mathcal{E}}_{\mathcal{E}\mathcal{E}} \to \infty$ as $n \to 1$; in the self-avoiding walk (SAW), the segments of loops strongly repel each other so that the process $\mathcal{E}\cdot\mathcal{E}\to I$ is suppressed.

Conformal perturbation theory
Epsilon-expansion around the Ising model (n=1)

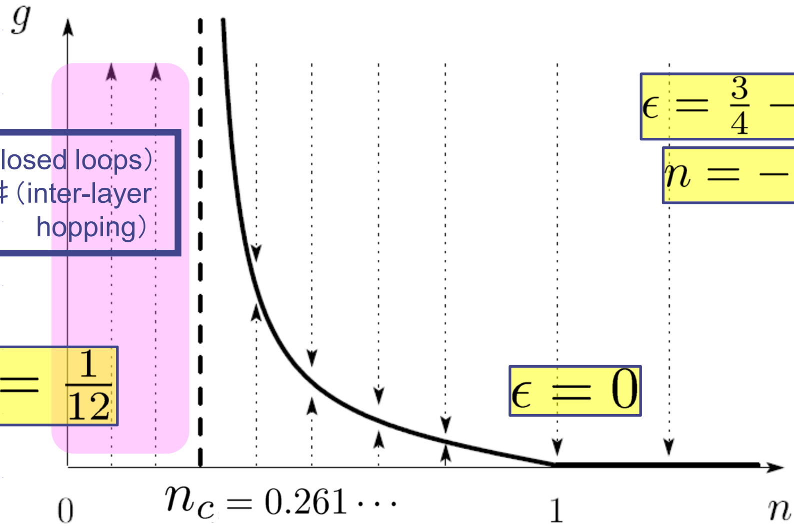
$$\overline{Z^M} = \mathrm{Tr}_{s_i^{(a)}} \exp\left[-\sum_{a=1}^{M} H_0^a - m_0 \int \sum_{a=1}^{M} \mathcal{E}^a(x) d^2x + g_0 \int \sum_{a\neq b}^{M} \mathcal{E}^a(x)\mathcal{E}^b(x) d^2x\right]$$

$$\mathcal{H}_0 = \sum_{a=1}^{M} H_0^a, \qquad \mathcal{H}_{int} = \int \sum_{a\neq b}^{M} \mathcal{E}^a(x)\mathcal{E}^b(x) d^2x.$$

Interlayer-coupling btw replicas

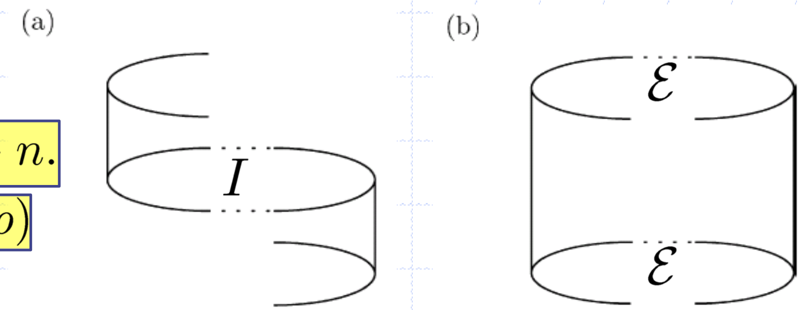
$$\epsilon \propto 1 - n.$$

$$\beta(g) = \frac{dg}{d\ln(r)} = 8\epsilon g + 2\pi\left[2(M-2) + \left(C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}}\right)^2\right] g^2.$$



$$\epsilon = \frac{3}{4} - \rho \propto 1 - n.$$

$$n = -2\cos(\pi/\rho)$$



$$\left(C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}}(\rho)\right)^2 = \left[2(1-2\rho)^2 \frac{\gamma^{\frac{3}{2}}(\rho)}{\gamma^2(2\rho)} \frac{\gamma^{\frac{1}{2}}(2-3\rho)}{\gamma(3-4\rho)}\right]^2, \sim 12^2 \cdot 0.01305 \cdot \epsilon^2.$$

a pole at $\epsilon = 1/12$

n=1: Kramars-Wannier duality

n=0: Infinite repulsions

- One-loop RG shows: non-trivial f.p. for $n_c < n < 1$ and strongly coupled phase for $n < n_c$

Diagrams at two-loop order

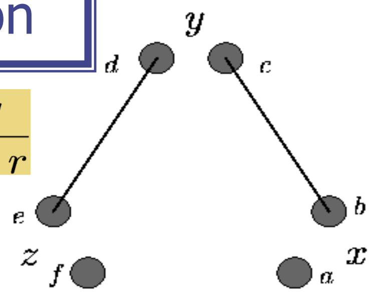
- Recall that the interaction is the coupling btw replicas:

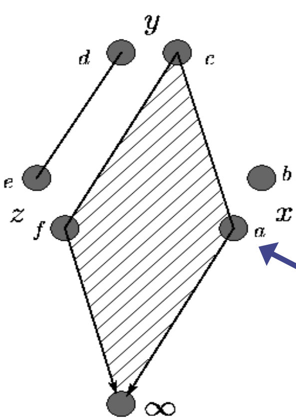
$$H_{int}(x) = \sum_{a\neq b}^{M} \mathcal{E}^a(x)\mathcal{E}^b(x).$$

- Beta-function, renormalization constants $Z_{\mathcal{E}}$ Z_{σ}

β-function

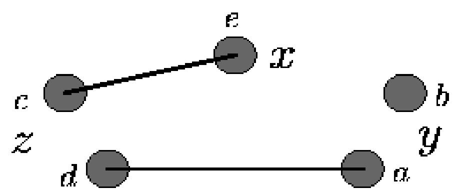
$$\beta(g) = \frac{dg}{d\ln r}$$

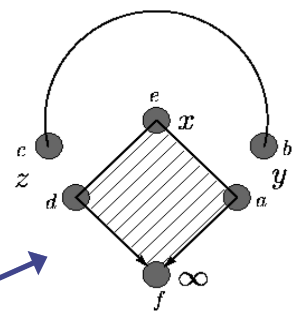




$Z_{\mathcal{E}}$

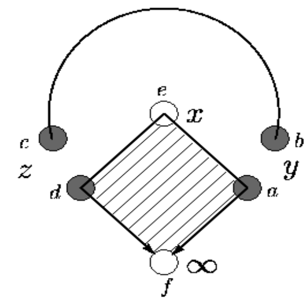
$$\gamma_{\mathcal{E}} = \frac{d\ln Z_{\mathcal{E}}}{d\ln r}$$

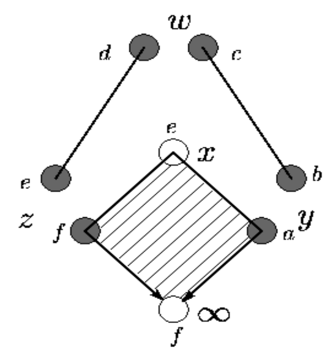




Z_{σ}

$$\gamma_{\sigma} = \frac{d\ln Z_{\sigma}}{d\ln r}$$





Boxes are four-point functions

$$\langle\mathcal{O}(0)\mathcal{O}(\infty)\rangle = \langle\mathcal{O}(0)\mathcal{O}(\infty)\rangle_0 + g_0\int_{|x|<r} d^2x\langle\mathcal{H}_{\rm int}(x)\mathcal{O}(0)\mathcal{O}(\infty)\rangle_0 + \frac{g_0^2}{2}\int_{|x|<r,\,|y|<r} d^2x d^2y\langle\mathcal{H}_{\rm int}(x)\mathcal{H}_{\rm int}(y)\mathcal{O}(0)\mathcal{O}(\infty)\rangle_0 + \ldots$$

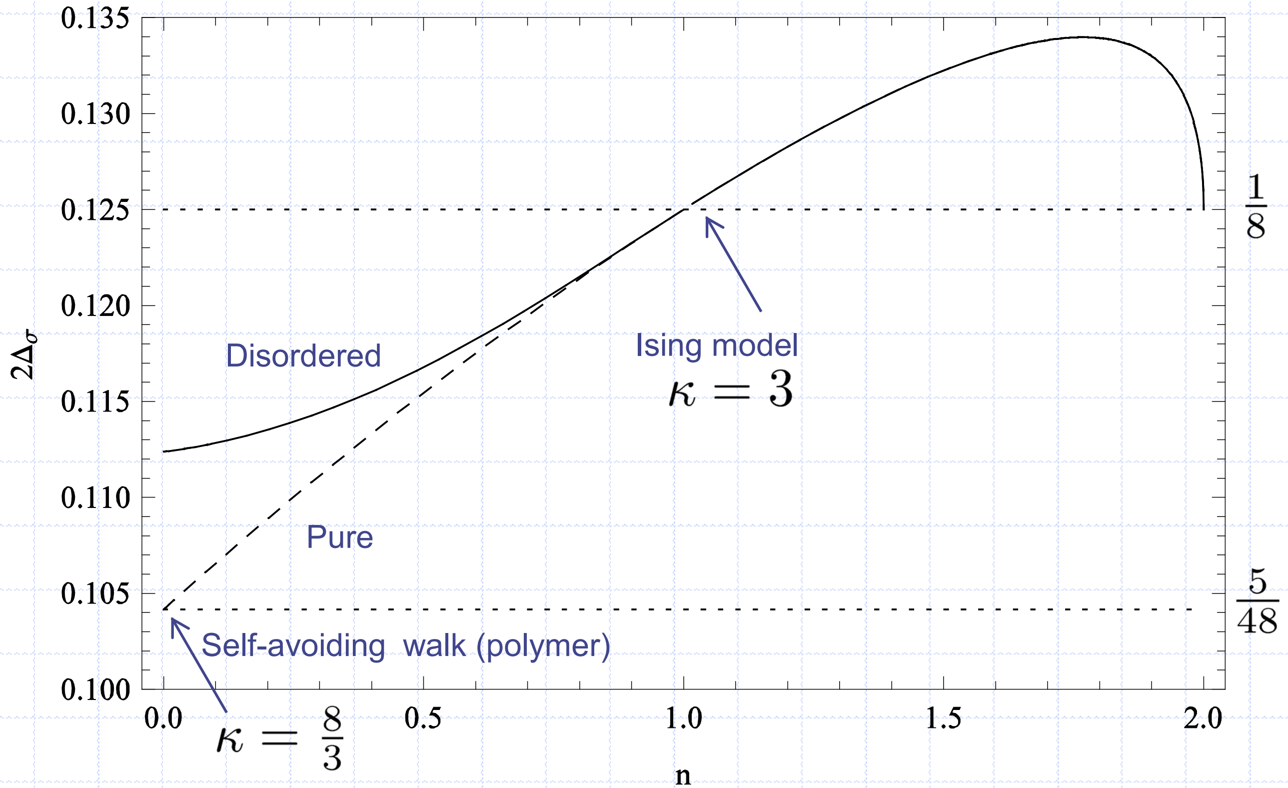
$$\mathcal{O}\to\mathcal{O}(1 + A_1 g_0 + A_2 g_0^2 + A_3 g_0^3 + \cdots) \equiv Z_{\mathcal{O}}\mathcal{O}.$$

The insertions of the interaction are restricted on the disk of radius r.

Spin Scaling Dimensions at the Non-trivial Fixed Point

For $\epsilon_\kappa = 3 - \kappa > 0$ with the SLE parameter κ given by $n = -2\cos\frac{4\pi}{\kappa}$,

$$2\left(\Delta_\sigma^{\mathrm{IR}} - \Delta_\sigma^{\mathrm{UV}}\right)_{O(n)} = \frac{\Gamma^4(\frac{1}{4})}{8\pi^4}\epsilon_\kappa^3 + \mathcal{O}\left(\epsilon_\kappa^4\right).$$



$$\frac{\Gamma^4(\frac{1}{4})}{8\pi^4} = 0.221735\cdots$$

About **+8%** shift in the spin dimension at the polymer point .
$\left(\epsilon_\kappa = \frac{1}{3}\right)$

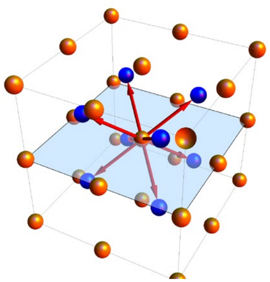
Number-theoretic character of the spin dimensions

H. Shimada, Nucl. Phys. B820 (2009) 707.

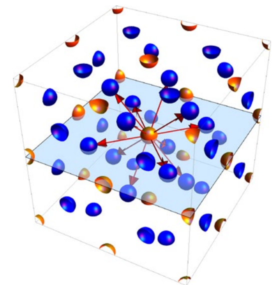
$$2\frac{K^2(\sin\frac{\pi}{4})}{\pi^3} = \frac{1}{2\pi}\left(\frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{du\,dv\,dw}{1-\cos u\cos v\cos w}\right) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\left(\frac{{}_{2n}C_n}{2^{2n}}\right)^3 = \frac{\Gamma\left(\frac{1}{4}\right)^4}{8\pi^4}$$

Vl.S. Dotsenko, M.Picco and P.Pujol, Nucl. Phys. **B455** (1995) 701.

$$\frac{2}{81\pi^2}\left[K\left(\sin\frac{\pi}{12}\right)K\left(\cos\frac{\pi}{12}\right)\right]^2 = \frac{27}{32}\frac{\Gamma\left(-\frac{2}{3}\right)^2\Gamma\left(\frac{1}{6}\right)^2}{\Gamma\left(-\frac{1}{3}\right)^2\Gamma\left(-\frac{1}{6}\right)^2} = \frac{2\pi^2}{81}\left(\frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{du\,dv\,dw}{3-\cos u\cos v-\cos v\cos w-\cos w\cos u}\right)^2$$



Number of Returns to the Origin
for the Random Walkers on the
BCC and FCC Lattices in an Infinite Time.

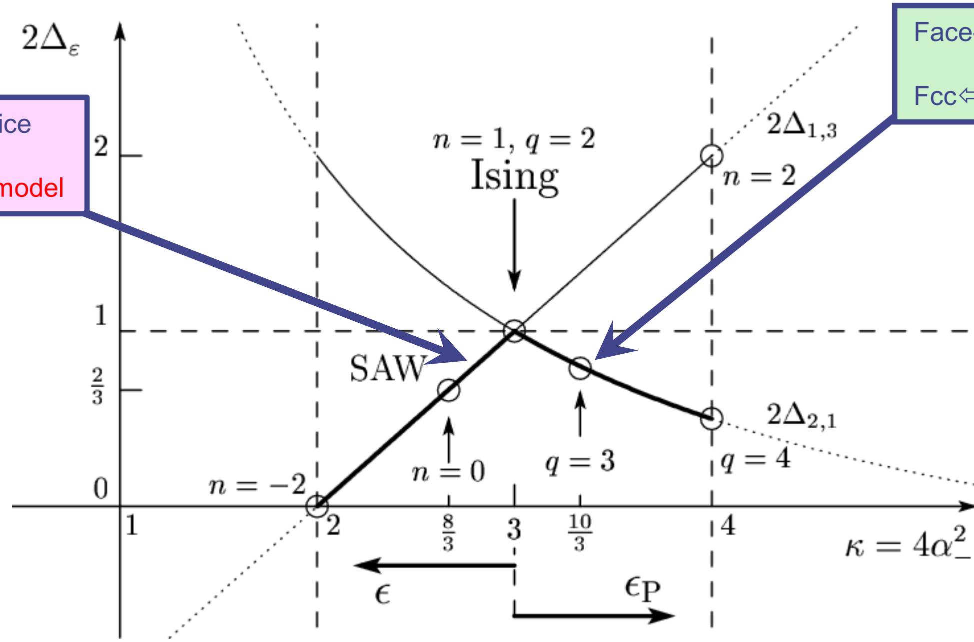


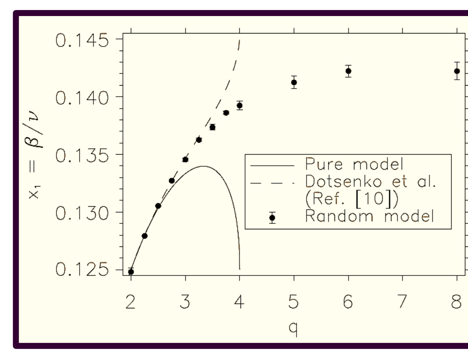
Body-centered cubic lattice

Bcc⇔ Disordered O(n) model

Face-centered cubic lattice

Fcc⇔ random-bond q-state Potts model





J. Cardy and J. L. Jacobsen, PRL 79 (1997) 4063.

These curious numbers characterize the two universal family of the disordered models
(manifestation of internal symmetries?)

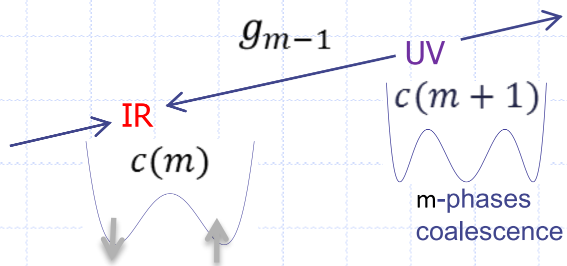
C-theorem and (non-)unitary systems

◆ C-theorem (valid for unitary models):

$$C(m) = 1 - \frac{6}{m(m+1)} \quad (m \geq 3)$$

Landau-Ginzburg formulation of the minimal models

$$\mathcal{L} = (\partial\varphi)^2 + \cdots + g_{m-1}\varphi^{2(m-1)}$$



$$C(m+1) - C(m) = \frac{12}{m(m+1)(m+2)} > 0$$

Ising model: $m = 3, C = \frac{1}{2}$, Tricritical Ising: $m = 4, C = \frac{7}{10}$

$$\vec{\beta} \sim -\nabla C$$

RG flow C-function

Concretely, along the RG flow,

$$\frac{1}{2}\beta^i \frac{\partial}{\partial g^i} C = -\frac{3}{4}(2\pi)^2 \mathcal{G}_{ij}\beta^i\beta^j.$$

degree of broken conformal symmetry

◆ In non-unitary systems, their "Zamolodchikov-metric" are not necessarily positive definite!

In fact, in the replica limit (M->0), the metric has a negative eigenvalue in the direction of the randomness.

$$\mathcal{G}_{ij} = (z\bar{z})^2 \langle \Phi_i(z,\bar{z})\Phi_j(0,0)\rangle\Big|_{z\bar{z}=1} \sim 2M(M-1).$$

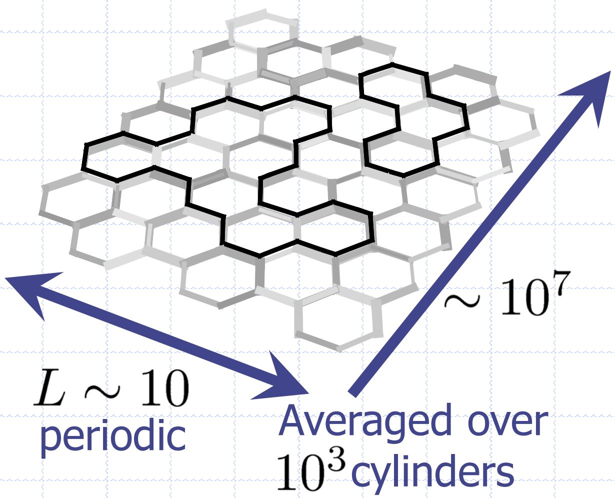
M : number of replicas

randomness coupling: $\Phi = \mathcal{H}_{\rm int} = \sum_{a\neq b}^{M} \mathcal{E}^{(a)}(x)\mathcal{E}^{(b)}(x)$

◆ RG flow is uphill in the randomness direction!

e.g.
Ludwig, Ludwig-Cardy 1987,
Fujita-Hikida-Ryu-Takayanagi 2008, H. Shimada 2009

Finite-size-scaling of the averaged free energies from Random Transfer Matrices



$$Z\left[\{x\},n\right] = \mathrm{Tr}_{s_i} \prod_{\langle i,j \rangle} (1 + x_{ij} s_i \cdot s_j)$$

$$P(x_{ij}) = \left[p\delta(x_{ij} - x_1) + (1-p)\delta(x_{ij} - x_2)\right].$$

Temperature : $t = 2x_c/(x_1 + x_2)$ $(p = 1/2)$

Randomness: $s^2 = x_2/x_1$

[Pure critical point: $(t, s) = (1, 1)$]

- The C-function can be extracted as the effective central charge using the finite-size scaling of the free energy for the cylinder of perimeter L:

$$C \leftrightarrow C_{\mathrm{eff}}$$

$$\overline{f(L)} = \overline{f(\infty)} - \frac{2}{\sqrt{3}} \frac{\pi C_{\mathrm{eff}}}{6L^2} + \mathcal{O}\left(\frac{1}{L^4}\right).$$

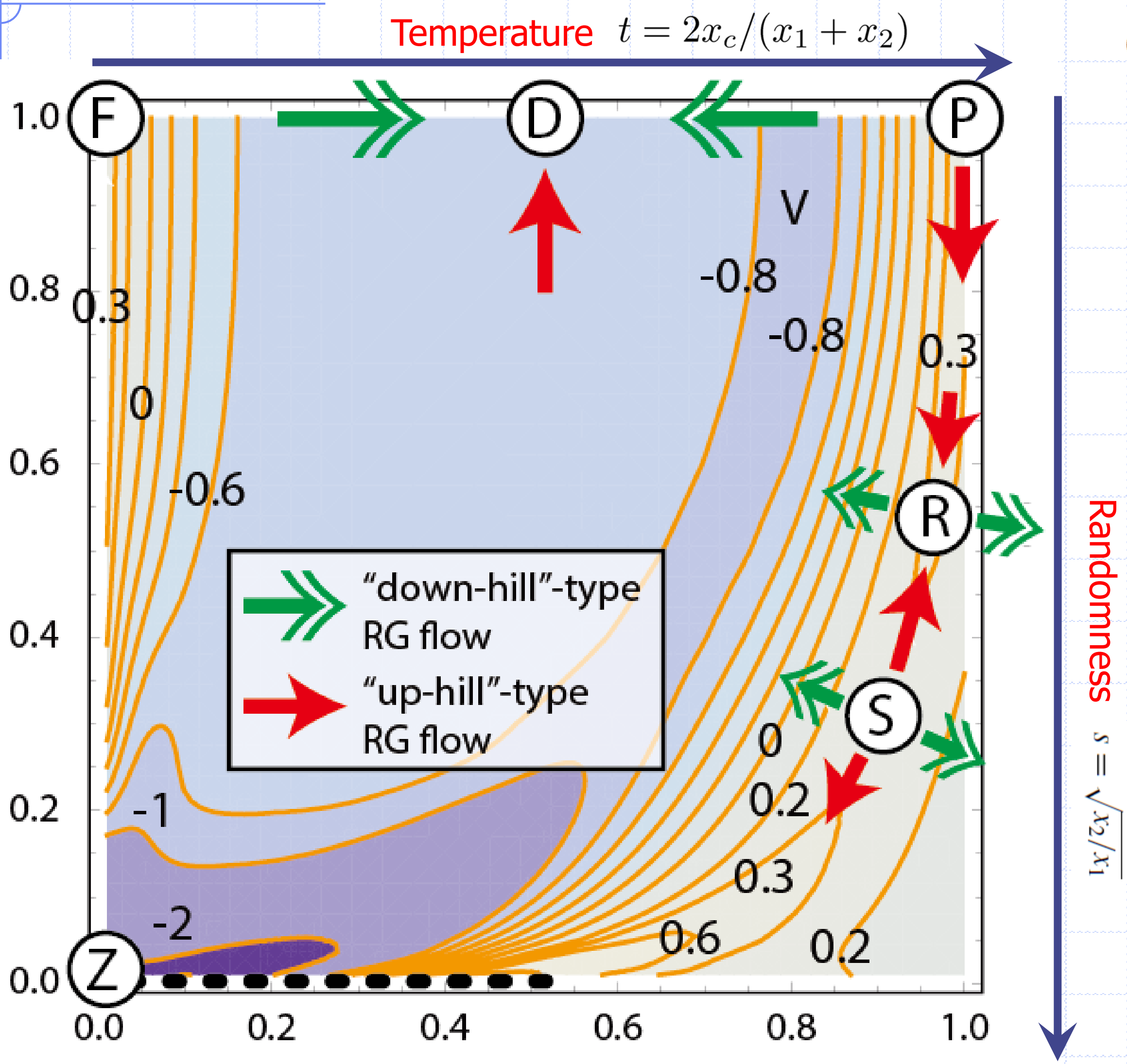
$$\frac{1}{2}\beta^i \frac{\partial}{\partial g^i} C = -\frac{3}{4}(2\pi)^2 \, \mathcal{G}_{ij}\beta^i\beta^j.$$

$\vec{\beta}$: RG flow

$$C_{\mathrm{eff}} = \left.\frac{\partial}{\partial M}\right|_{M=0} c_M$$

RG landscape for the disordered O(n) model

$n = 0.6$

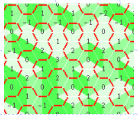


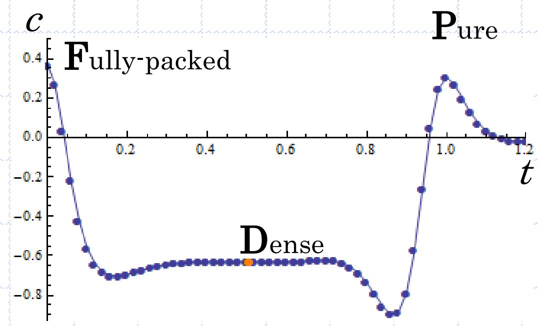
H.Shimada, J.L.Jacobsen, Y. Kamiya, J. Phys. A 47 (2014) 122001.

Fully-packed loop model: symmetry is enhanced

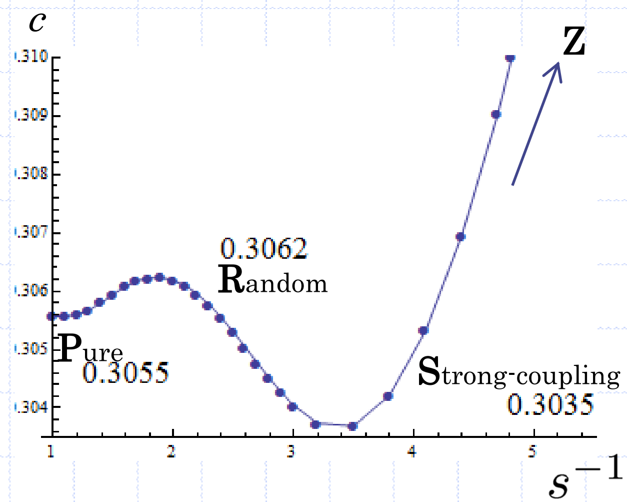
c(F) = c(D) + 1.

de Gier-Kondev-Nienhuis 96





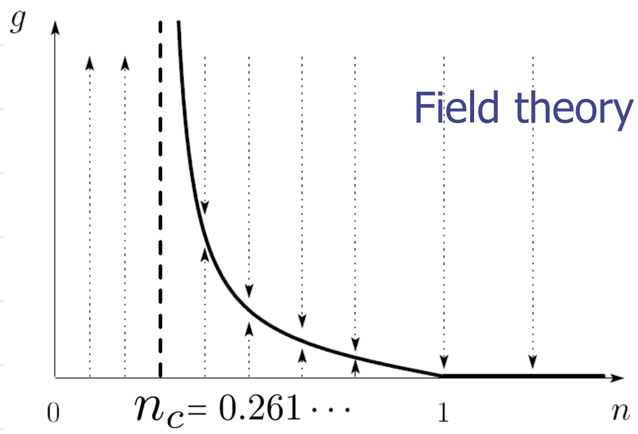
Along the pure line s=1



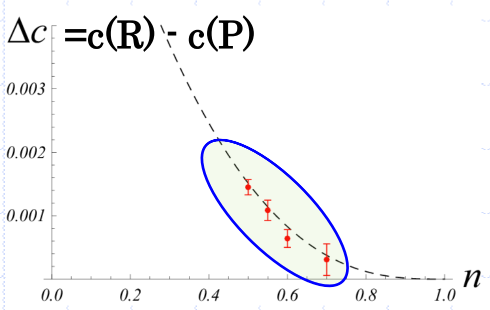
Along the ridge (critical line)

The multicritical point S is strongly-coupled

Randomness



Transfer Matrix



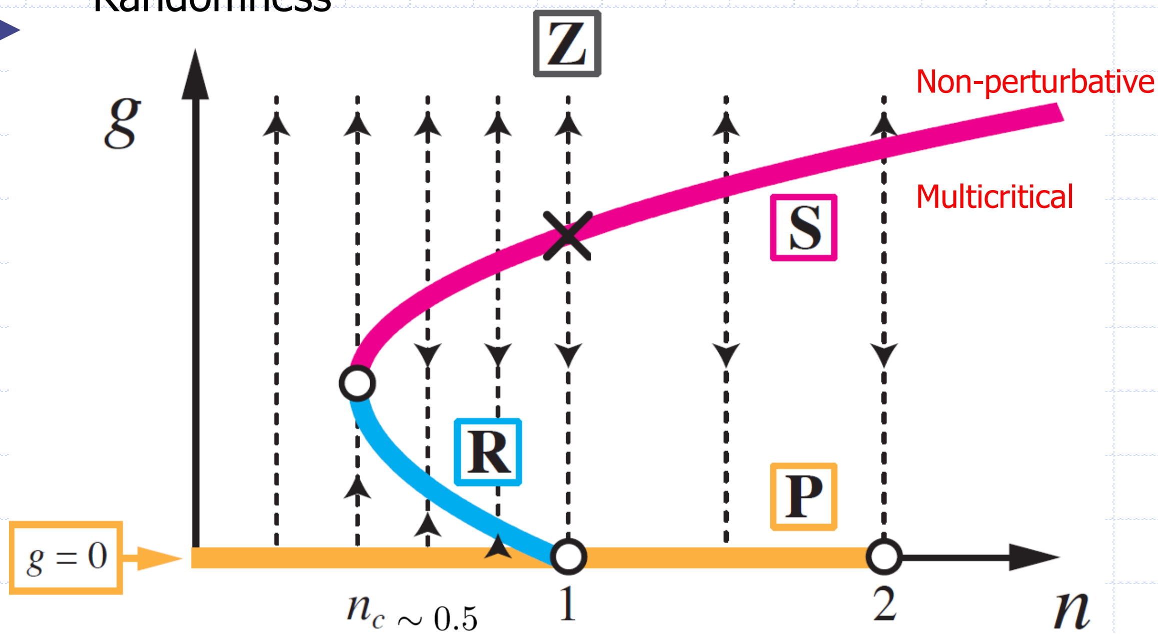
- The fixed point **S** appears to be in the non-perturbative regime since the Borel-Pade analysis of n=1 RG beta function precludes IR FPs ($M < M_c \sim \frac{11}{4}$) with a finite coupling g>0.

$$\mathcal{L} = \bar{\psi}_a \partial \psi^a + g(\bar{\psi}_a \psi^a)^2 \qquad (a = 1,2,\cdots,M)$$

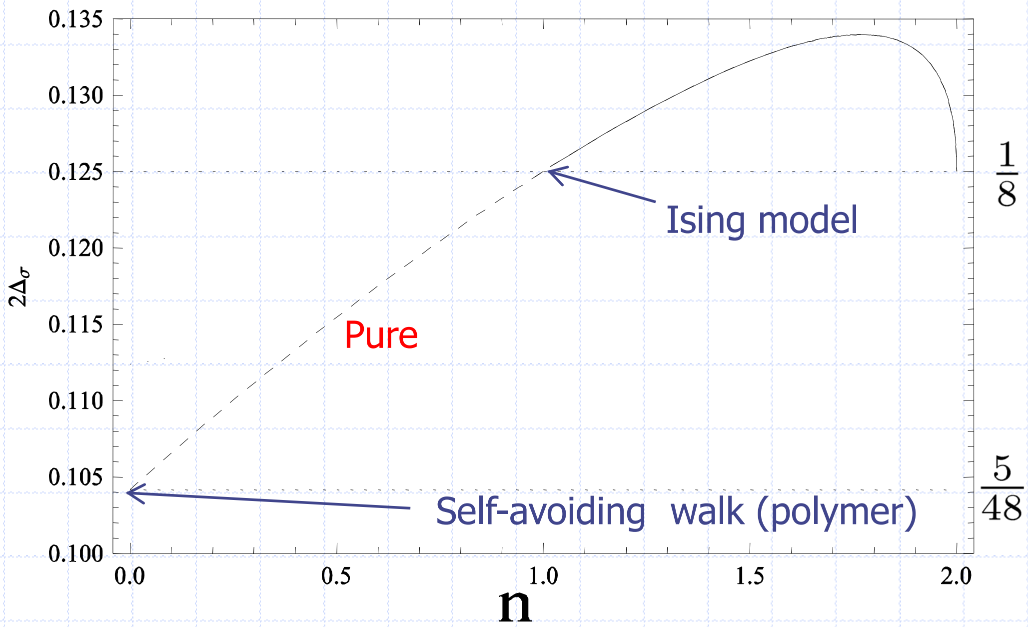
$$\beta(g) = (M-2)\left[g^2 - g^3 - \frac{(M-7)}{4} g^4\right] + \mathcal{O}(g^5)$$

M-color Gross-Neveu model at 3-loop (Gracey 91)

Randomness



Spin Scaling Dimension of the weakly-coupled FP $\mathbf{R}$

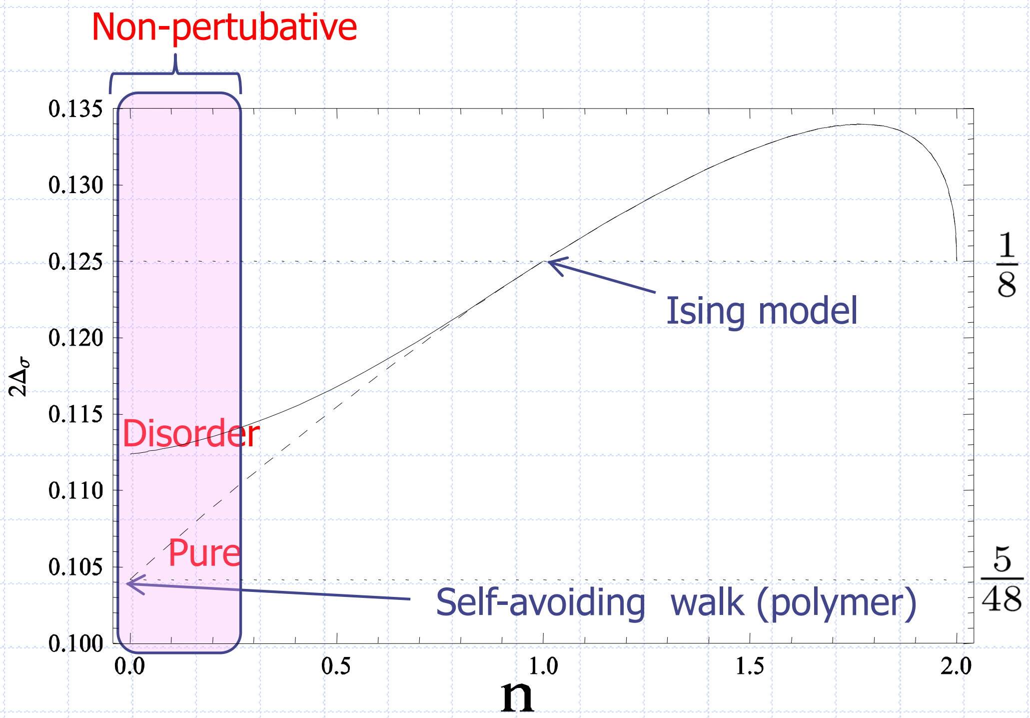


The spin scaling dimension in the O(n) model

$$2\Delta_\sigma^{\mathrm{P}} = 1 - \frac{\rho}{2} - \frac{3}{8\rho}. \qquad n = -2\cos\pi/\rho$$

(Conjecture) 80 Cardy-Hamber
(Coulomb gas) 82 Nienhuis
(CFT) 84 Dotsenko-Fateev
(Bethe ansatz) 88 Batchelor-Blote

Spin Scaling Dimension of the weakly-coupled FP $\mathbf{R}$



The spin scaling dimension in the O(n) model

$$2\Delta_\sigma^{\mathrm{P}} = 1 - \frac{\rho}{2} - \frac{3}{8\rho}. \qquad n = -2\cos\pi/\rho$$

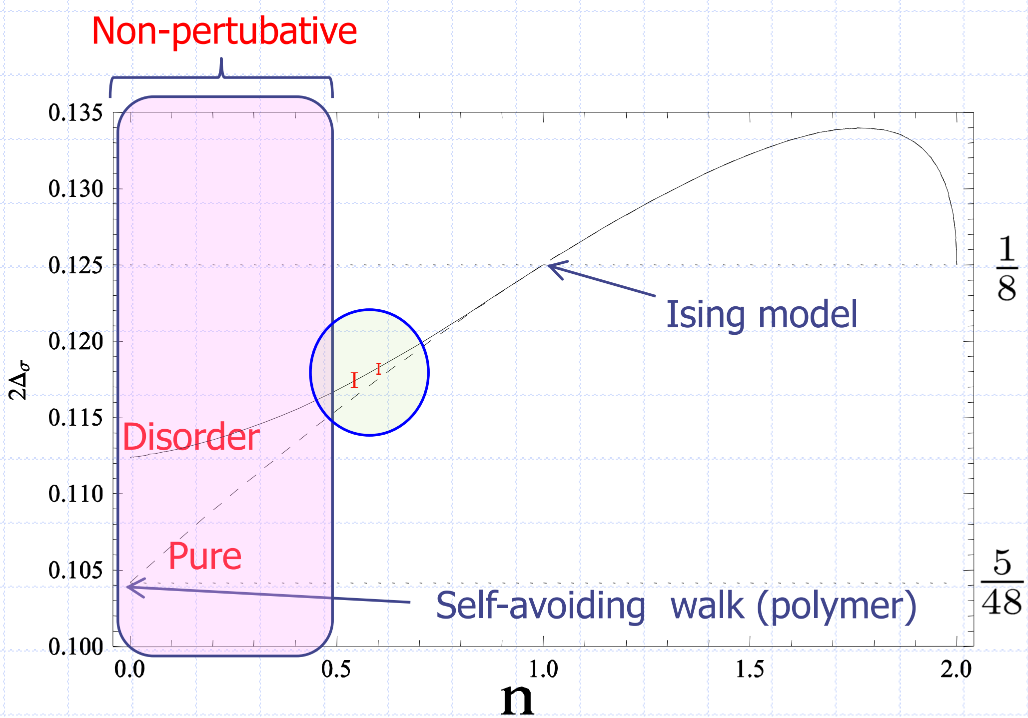
(Conjecture) 80 Cardy-Hamber
(Coulomb gas) 82 Nienhuis
(CFT) 84 Dotsenko-Fateev
(Bethe ansatz) 88 Batchelor-Blote

In the presence of Randomness ,

$$2\left(\Delta_\sigma^{\mathrm{R}} - \Delta_\sigma^{\mathrm{P}}\right)_{O(n)} = \frac{8\Gamma^4(\frac{1}{4})}{\pi^4}\epsilon_\rho^3 + \mathcal{O}\left(\epsilon_\rho^4\right).$$

$$\frac{8\Gamma^4(\frac{1}{4})}{\pi^4} = 1.7738\cdots \qquad \epsilon_\rho = \frac{3}{4} - \rho$$

Spin Scaling Dimension of the weakly-coupled FP $\mathbf{R}$



The spin scaling dimension in the O(n) model

$$2\Delta_\sigma^{\mathrm{P}} = 1 - \frac{\rho}{2} - \frac{3}{8\rho}. \qquad n = -2\cos\pi/\rho$$

(Conjecture) 80 Cardy-Hamber
(Coulomb gas) 82 Nienhuis
(CFT) 84 Dotsenko-Fateev
(Bethe ansatz) 88 Batchelor-Blote

In the presence of Randomness ,

$$2\left(\Delta_\sigma^{\mathrm{R}} - \Delta_\sigma^{\mathrm{P}}\right)_{O(n)} = \frac{8\Gamma^4(\frac{1}{4})}{\pi^4}\epsilon_\rho^3 + \mathcal{O}\left(\epsilon_\rho^4\right).$$

$$\frac{8\Gamma^4(\frac{1}{4})}{\pi^4} = 1.7738\cdots \qquad \epsilon_\rho = \frac{3}{4} - \rho$$

	$n = 0.55$	$n = 0.60$
Disorderd model (Transfer matrix)	0.1174±0.0004	0.11826±0.0002
Disorderd model (Field theory)	0.11748	0.11822
Pure model (Exact formula)	0.11647	0.11749

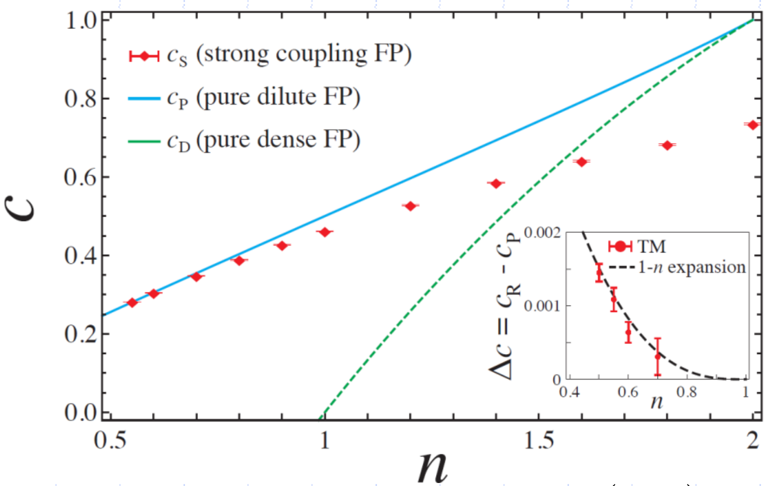
TM results confirm the FT predictions!

$\mathbf{S}$ at $n = 1$ is in the Nishimori universality class

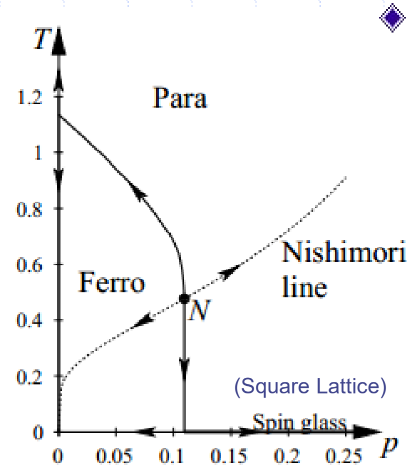
At n=1, $\mathbf{S}$ has the central charge $c = 0.4612 \pm 0.0004$, which lies inside the error bar of $c = 0.464 \pm 0.004$ (A.Honecker et al., 01PRL) found for the Nishimori point in the $\pm$J random-bond Ising (Edwards-Anderson) model for spin glasses.

There, the symmetry $\pm$J leads to the local $\mathbb{Z}_2$ gauge invariance or supersymmetry osp(2m+1|2m).

H. Nishimori, Prog. Theor. Phys. **66**, 1169 (1981). I.A. Gruzberg, A.W.W. Ludwig, and N. Read, PRB **63** (2001) 104422



Shimada-Jacobsen-Kamiya, J. Phys. A (2014)



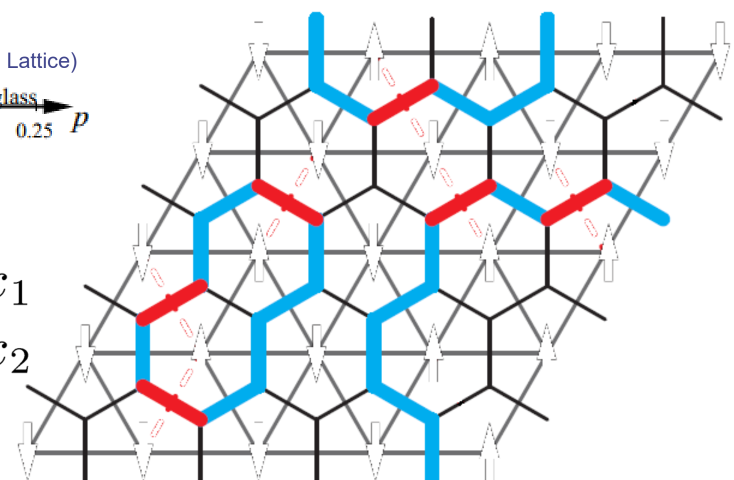
- By considering the honeycomb loops as two-type domain walls, n=1 is dual to the Ising spin-glass on the triangular lattice.

$$P(x) = p\delta(x - x_1) + (1-p)\delta(x - x_2),$$

$$t = \gamma\left[px_1 + (1-p)x_2\right]^{-1}, \quad s = \sqrt{x_2/x_1}$$

Red: x_1

Blue: x_2

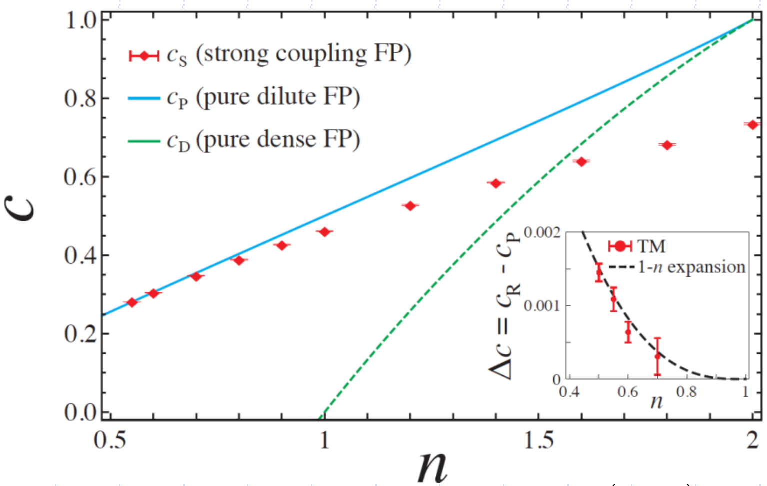


S at $n = 1$ is in the Nishimori universality class

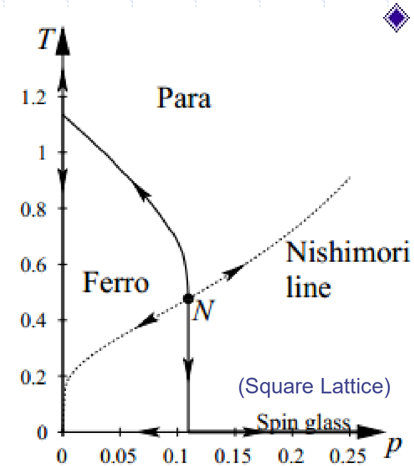
At n=1, **S** has the central charge $c = 0.4612 \pm 0.0004$, which lies inside the error bar of $c = 0.464 \pm 0.004$ (A.Honecker et al., 01PRL) found for the Nishimori point in the ±J random-bond Ising (Edwards-Anderson) model for spin glasses.

There, the symmetry ±J leads to the local $\mathbb{Z}_2$ gauge invariance or supersymmetry osp(2m+1|2m).

H. Nishimori, Prog. Theor. Phys. **66**, 1169 (1981). I.A. Gruzberg, A.W.W. Ludwig, and N. Read, PRB **63** (2001) 104422



Shimada-Jacobsen-Kamiya, J. Phys. A (2014)



- By considering the honeycomb loops as two-type domain walls, n=1 is dual to the Ising spin-glass on the triangular lattice.

$$x_i = \exp(2\beta J_i), \quad i = 1, 2$$

Disordered Loop Spin glass

Location of **S**

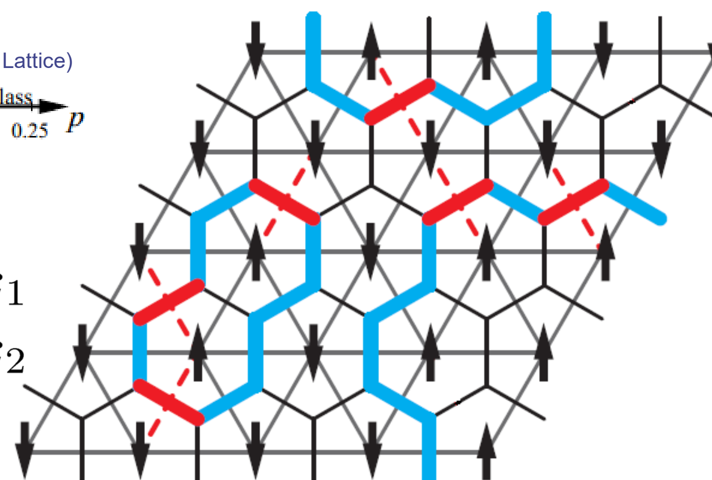
$$p = 0.5, J_2/J_1 \sim -9.64$$

$$p = 0.1639\cdots, J_2/J_1 = -1$$ (±J –spin glass)

agrees with the multicritical pt. in the Triangular Lattice-RBIM e.g. M.Ohzeki 08~

Red: x_1

Blue: x_2

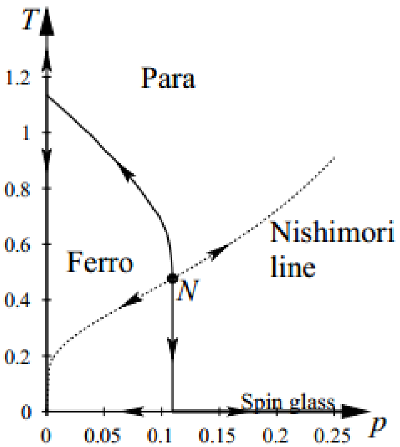


$\mathbf{S}$ at $n = 1$ is in the Nishimori universality class

At n=1, $\mathbf{S}$ has the central charge $c = 0.4612 \pm 0.0004$, which lies inside the error bar of $c = 0.464 \pm 0.004$ (A.Honecker et al., 01PRL) found for the Nishimori point in the ±J random-bond Ising (Edwards-Anderson) model for spin glasses.

There, the symmetry ±J leads to the local $\mathbb{Z}_2$ gauge invariance or supersymmetry osp(2m+1|2m).

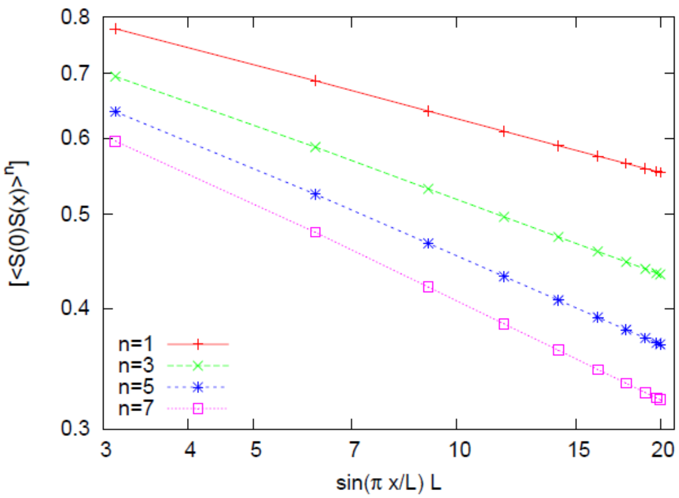
H. Nishimori, Prog. Theor. Phys. **66**, 1169 (1981). I.A. Gruzberg, A.W.W. Ludwig, and N. Read, PRB **63** (2001) 104422



(Square Lattice)

$$\overline{\langle S(x_1)S(x_2)\rangle^n} \propto |x_1 - x_2|^{-\eta_n}$$

$$\eta_1 = \eta_2 = 0.1848 \pm 0.0003,$$
$$\eta_3 = \eta_4 = 0.2552 \pm 0.0009,$$
$$\eta_5 = \eta_6 = 0.3004 \pm 0.0013,$$
$$\eta_7 = \eta_8 = 0.3341 \pm 0.0016.$$



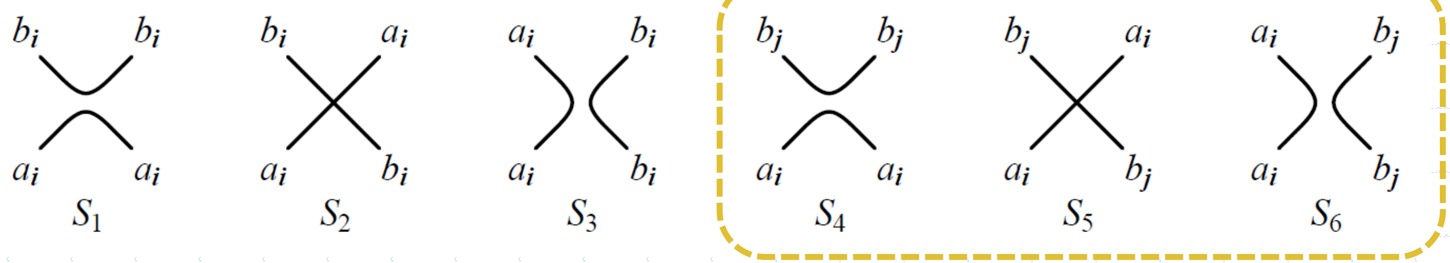
M. Picco, A. Honecker, and P. Pujol, J. Stat. Mech. (2006) P09006.

A. Honecker, M. Picco, and P. Pujol, PRL 87 (2001) 047201.

(Merz-Chalker 2002, parabolic spectrum of (dual) disorder operator moments)

New exact approach from S-matrix: Disordered O(n) model as an example

G.Delfino, Eur. Phys. J. B **94** (2021) 65
Particles, conformal invariance and criticality in pure and disordered systems
G.Delfino and N. Lamsen, JHEP 04 (2018) 077, J. Stat. Mech. (2019) 024001

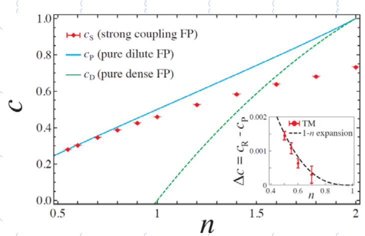


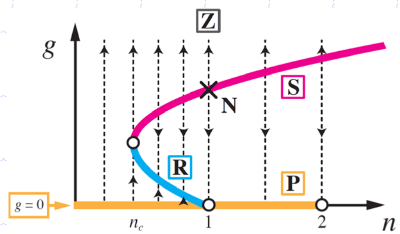
Interlayer coupling

$$
\begin{aligned}
S_1 = S_3^* &\equiv \rho_1\, e^{i\phi}, \\
S_2 = S_2^* &\equiv \rho_2, \\
S_4 = S_6^* &\equiv \rho_4\, e^{i\theta}, \\
S_5 = S_5^* &\equiv \rho_5\,.
\end{aligned}
$$

$$
\begin{aligned}
&\rho_1^2 + \rho_2^2 = 1 \\
&\rho_1\rho_2 \cos\phi = 0, \\
&n\rho_1^2 + n(M-1)\rho_4^2 + 2\rho_1\rho_2\cos\phi + 2\rho_1^2\cos 2\phi = 0 \\
&\rho_4^2 + \rho_5^2 = 1, \\
&\rho_4\rho_5\cos\theta = 0, \\
&2n\rho_1\rho_4\cos(\phi-\theta) + n(M-2)\rho_4^2 + 2\rho_2\rho_4\cos\theta + 2\rho_1\rho_4\cos(\phi+\theta) = 0
\end{aligned}
$$

$$\sum_{e,f} S_{ab}^{ef}\left[S_{ef}^{cd}\right]^* = \delta_{ac}\delta_{bd}$$

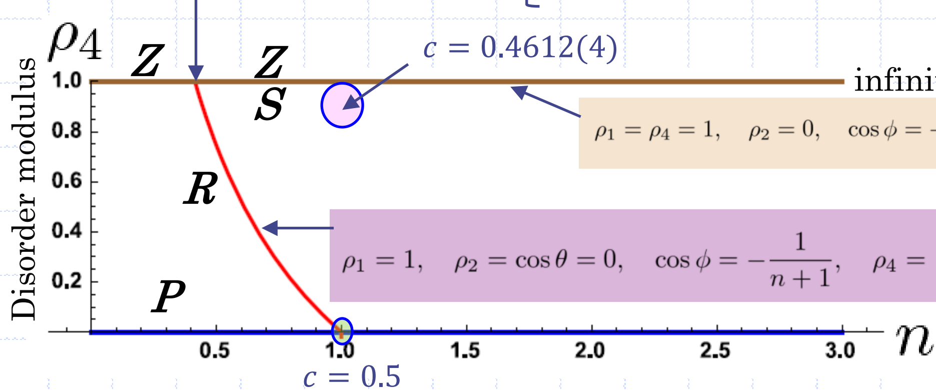




$n_c = \sqrt{2} - 1 \approx 0.41$ (exact)

$n_c \approx 0.5$ (transfer matrix)
$n_c \approx 0.26$ (one-loop RG)

$M = 0$
IR FPs
Modified from the original



$\rho_1 = \rho_4 = 1, \quad \rho_2 = 0, \quad \cos\phi = -\frac{1}{\sqrt{2}}, \quad \cos\theta = -\frac{n^2+2n-1}{\sqrt{2}\,(n^2+1)}$

$\rho_1 = 1, \quad \rho_2 = \cos\theta = 0, \quad \cos\phi = -\frac{1}{n+1}, \quad \rho_4 = \frac{1-n}{1+n}\sqrt{\frac{n+2}{n}}$

Epsilon expansion from Ising CFT (d=2 fixed) varying the size of internal symmetry

In 2d, the Ising model saturates the Harris criterion.
The random-bond Ising model (RBIM) plays a pivotal role.

$$\sum_{a\neq b}^{M} \mathcal{E}^a(x)\mathcal{E}^b(x)$$ $2\Delta_\varepsilon = 2$ (marginal)

Randomness is relevant if $\epsilon \propto 2 - 2\Delta_\varepsilon > 0$

Ising model = "O($n = 1$) model" = "$q = 2$-state Potts model"

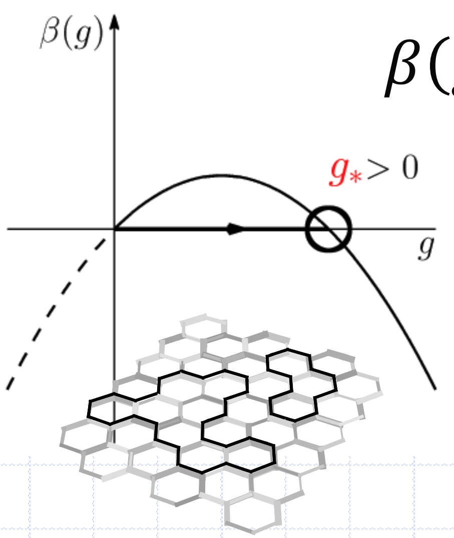
$(q-2)$–expansion

A.W.W. Ludwig, Nucl. Phys. B285 (1987) 97.

Small deviation from the Ising model: $\epsilon \propto (1-n)$ or $(q-2)$
RG beta function $\beta(g) = \epsilon g + [(M-2) + C^2(\epsilon)]g^2 + O(g^3)$ (schematic)

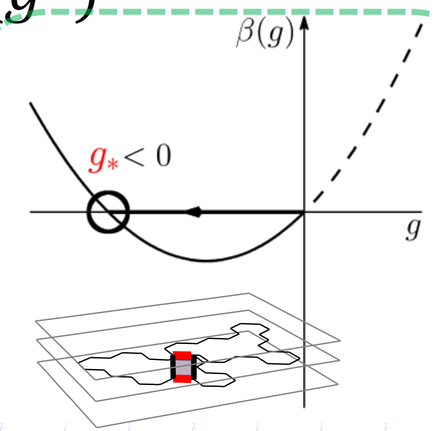
$C(\epsilon) = C_{\varepsilon\varepsilon}^{\varepsilon}(\epsilon)$ vanishes for the 2d Potts critical points (self-duality)

$$\beta(g) = \epsilon g + (M-2)g^2 + O(g^3)$$



IR fixed point is located at

$$g_* = \frac{\epsilon}{2-M} + O(\epsilon^2)$$



M-coupled q-state Potts models (Unitary) with $M = 3,4,5,\cdots$

Disordered CFTs @ $M \to 0$ (Non-unitary)

$M = 2, q > 2$ is integrable and massive (Vaysburd 95)

M-coupled q-state Potts models with M=3,4,5,⋯

$$H = \sum_{\langle ij \rangle} \mathcal{H}_{ij}$$

$$\mathcal{H}_{ij} = -\sum_{m=1}^{M} K_m \sum_{1\leqslant\mu_1<\cdots<\mu_m\leqslant M} \prod_{l=1}^{m} \delta_{\sigma_i^{(\mu_l)}\sigma_j^{(\mu_l)}}$$

M=3 coupled q-state Potts model

Dotsenko-Jacobsen-Lewis-Picco, Nucl. Phys. B 546 (1999) 505.
"Coupled Potts models: Self-duality and fixed point structure"

$$\mathcal{H}_{ij} = -\left[K_1\left(\delta_{\sigma_i\sigma_j} + \delta_{\tau_i\tau_j} + \delta_{\eta_i\eta_j}\right) + K_2\left(\delta_{\sigma_i\sigma_j}\delta_{\tau_i\tau_j} + \delta_{\tau_i\tau_j}\delta_{\eta_i\eta_j} + \delta_{\sigma_i\sigma_j}\delta_{\eta,\eta'}\right) + K_3\delta_{\sigma_i\sigma_j}\delta_{\tau_i\tau_j}\delta_{\eta_i\eta_j}\right]$$

(1) $K_1 \neq 0,\ K_2 = K_3 = 0$: 3-decoupled q-state Potts FP
(2) $K_1 = K_2 = 0,\ K_3 \neq 0$: Single q^3-state Potts FP ("massive" if $q^3 > 4$)
(3) $K_1 \neq 0,\ K_2 \neq 0,\ K_3 \neq 0$: Non-trivial FP ($S_3$-extended q-state Potts CFT)

q	$c_{\rm pure}$	$c_{\rm FP}$
2.00	1.5000	1.5000
2.25	1.7627	1.7620
2.50	1.9975	1.9931
2.75	2.2089	2.1976
3.00	2.4000	2.3808
3.25	2.5734	2.5500
3.50	2.7309	2.7164
3.75	2.8734	2.9054
4.00	3.0000	3.3750

$\alpha_+^2 = \frac{4}{3} - \epsilon \qquad \sqrt{q} = 2\cos\frac{\pi}{p+1}, \alpha_+^2 = \frac{p+1}{p}$

$\epsilon = 2/15$ for $q = 3$ state Potts model

$$\Delta c = -24\int_0^{g_*} \beta(g)\mathrm{d}g = -\frac{27}{8}\frac{M(M-1)}{(M-2)^2}\left(\epsilon^3 - \frac{9}{2(M-2)}\epsilon^4\right) + \mathcal{O}\left(\epsilon^5\right)$$

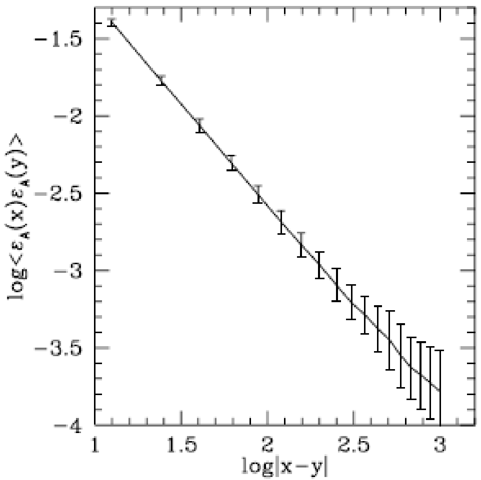
$c_{decoupled} = 0.8 \times 3 = 2.4$ for (1)

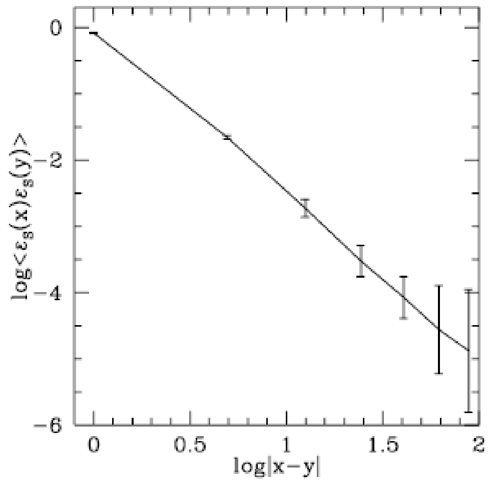
$c_{TM} = 2.377 \pm 0.003$
$c_{FT} = 2.3808 + O(\epsilon^5)$ for (3)

Various scaling behaviors established by Monte Carlo

Precision is not intended.

M=3





q	$\Delta_{\varepsilon_1+\varepsilon_2+\varepsilon_3}$	$\Delta_{\varepsilon_1-\varepsilon_2}$
2.00	1.0000	1.0000
2.25	1.1447	0.8499
2.50	1.2639	0.7154
2.75	1.3615	0.5930
3.00	1.4400	0.4800
3.25	1.5006	0.3737
3.50	1.5429	0.2710
3.75	1.5624	0.1656
4.00	1.5000	0.0000

$$\varepsilon_S \equiv \varepsilon_1+\varepsilon_2+\varepsilon_3$$
$$\varepsilon_A \equiv \varepsilon_1-\varepsilon_2$$

$$\Delta_{\varepsilon_S} \equiv 1.27 \pm 0.13$$
$$\Delta_{\varepsilon_A} \equiv 0.63 \pm 0.04$$

$$\Delta_{\varepsilon_1+\varepsilon_2+\varepsilon_3} = \Delta_\varepsilon + 6\epsilon - 9\epsilon^2 + \mathcal{O}\left(\epsilon^3\right)$$
$$\Delta_{\varepsilon_1-\varepsilon_2} = \Delta_\varepsilon - 3\epsilon + \frac{9}{2}\epsilon^2 + \mathcal{O}\left(\epsilon^3\right)$$

For results on more operators:

Dotsenko-Jacobsen-Lewis-Picco, Nucl. Phys. B 546 (1999) 505.

Spectrum of higher operators

M=3

(q-2)-expansion

q	$\Delta_{\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1}$	$\Delta_{\varepsilon_1\varepsilon_2-\varepsilon_2\varepsilon_3}$	$\Delta_{\varepsilon_1\varepsilon_2\varepsilon_3}$	Δ_{σ_1}	$\Delta_{\sigma_1\sigma_2}$	$\Delta_{\sigma_1\sigma_2\sigma_3}$
2.00	2.000	2.000	3.000	0.12500	0.2500	0.3750
2.25	2.005	1.834	2.837	0.12789	0.2775	0.4553
2.50	2.021	1.653	2.664	0.12964	0.2949	0.5190
2.75	2.046	1.456	2.479	0.12985	0.3030	0.5685
3.00	2.080	1.240	2.280	0.12805	0.3023	0.6048
3.25	2.126	0.997	2.060	0.12353	0.2921	0.6283
3.50	2.186	0.713	1.806	0.11501	0.2703	0.6375
3.75	2.272	0.350	1.486	0.09926	0.2303	0.6268
4.00	2.500	-0.500	0.750	0.04238	0.0975	0.5301

Transfer matrix

q	$x_H^{(1)}(8,10)$	$x_H^{(2)}(6,8)$	$x_H^{(3)}(6,8)$
2.00	0.125112	0.276866	0.471563
2.25	0.127851	0.287143	0.500930
2.50	0.129810	0.295985	0.529874
2.75	0.131050	0.303470	0.558541
3.00	0.131623	0.309661	0.587025
3.25	0.131577	0.314615	0.615388
3.50	0.130962	0.318393	0.643668
3.75	0.129831	0.321061	0.671885
4.00	0.128237	0.322693	0.700047

Odd sector

$(\varepsilon_S^2 \equiv \varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1,\ \varepsilon_A^2 \equiv \varepsilon_1\varepsilon_2-\varepsilon_2\varepsilon_3)$

$$\Delta_{\varepsilon_S^2} = 2\Delta_\varepsilon(\epsilon)+3\epsilon+\frac{9}{2}\epsilon^2+\mathcal{O}(\epsilon^3)$$

$$\Delta_{\varepsilon_A^2} = 2\Delta_\varepsilon(\epsilon)-\frac{3}{2}\epsilon-9\epsilon^2+\mathcal{O}(\epsilon^3)$$

$$\Delta_{\varepsilon_1\varepsilon_2\varepsilon_3} = 3\Delta_\varepsilon(\epsilon)-\frac{27}{4}\epsilon^2+\mathcal{O}(\epsilon^3)$$

$$\Delta_{\sigma_1} = \Delta_\sigma - \frac{27}{16}\frac{M-1}{(M-2)^2}\mathcal{F}\epsilon^3+\mathcal{O}(\epsilon^4) \qquad \mathcal{F}=2\frac{\Gamma^2\left(-\frac{2}{3}\right)\Gamma^2\left(\frac{1}{6}\right)}{\Gamma^2\left(-\frac{1}{3}\right)\Gamma^2\left(-\frac{1}{6}\right)}$$

$$\Delta_{\sigma_1\sigma_2} = 2\Delta_\sigma(\epsilon)+\frac{3\epsilon}{4(M-2)}\left(1-\frac{3\epsilon}{M-2}\left((M-2)\log 2+\frac{11}{12}\right)\right)+\mathcal{O}(\epsilon^3),$$

$$\Delta_{\sigma_1\sigma_2\sigma_3} = 3\Delta_\sigma(\epsilon)+\frac{9\epsilon}{4(M-2)}\left(1-\frac{3\epsilon}{M-2}\left((M-2)\log 2+\frac{11}{12}+\frac{\alpha}{24}\right)\right)+\mathcal{O}(\epsilon^3) \qquad \alpha = 33-\frac{29\sqrt{3}\pi}{3}$$

Even sector

Gap	$x(4)$	$x(6)$	$x(8)$	$x(10)$	$x(12)$	Extrapolation	Operator
1	1.694	1.471	1.385	1.344	1.322	1.27	$\varepsilon_1+\varepsilon_2+\varepsilon_3$
2	2.603	2.380	2.272	2.219	2.148	≈ 2.0	$T=L_{-1}I$
3	2.922	2.775	2.398	2.225	2.148	≈ 2.0	$\overline{T}=\overline{L}_{-1}I$
4	—	3.104	2.398	2.225	2.174	≈ 2.1	$\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1$
5	—	3.104	2.685	2.633	2.599	≈ 2.3	$L_{-1}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$
6	—	3.626	3.279	2.785	2.598	≈ 2.3	$\overline{L}_{-1}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$
7	—	3.991	3.279	2.785	2.598	≈ 2.4	$\varepsilon_1\varepsilon_2\varepsilon_3$
8	—	3.991	3.289	3.146	3.066	≈ 3.0	$T'=L_{-3}I$

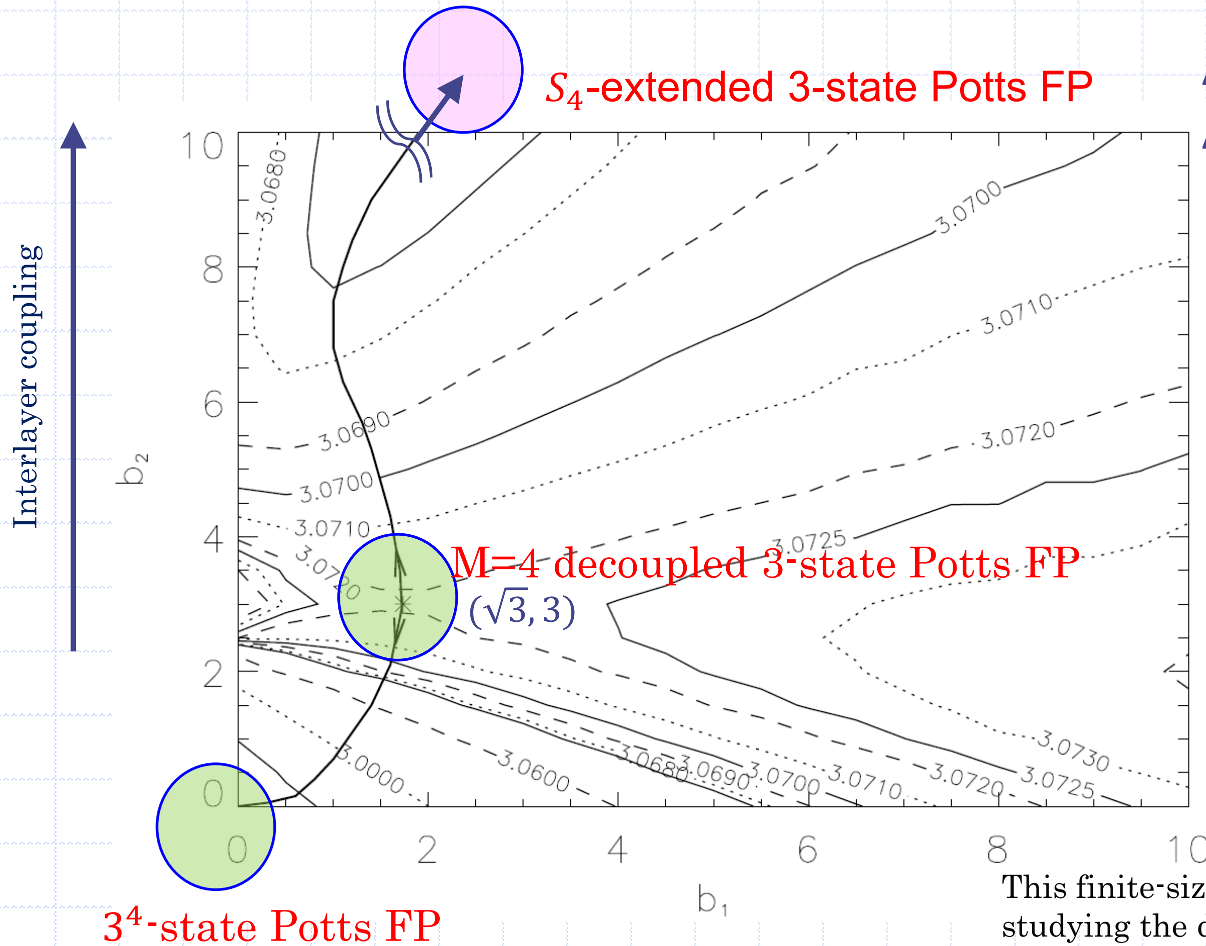
q=3

Dotsenko-Jacobsen-Lewis-Picco, Nucl. Phys. B 546 (1999) 505.

RG flow on the self-dual manifold of couplings

J. L. Jacobsen, Phys. Rev. E62 (2000) 1. "Duality relations for M coupled Potts models"

- Established explicit duality transformations. The self-dual manifold has dimension [M/2].
- Found the non-trivial FP using the C-function landscape obtained by the transfer matrix (TM).



$\Delta c_{FT} = -0.0168 + O(\epsilon^5)$

$\Delta c_{TM} \approx -0.02 \sim -0.01$

$c_{\text{decoupled; TM}} = 3.072$ is smaller than $c_{\text{decoupled;exact}} = 0.8 \times 4 = 3.2$ by 4%

This finite-size correction (4%) is not an issue when studying the difference Δc_{TM} of the central charge if it is assumed to affect all the FPs almost equally.

Conclusion and outlooks

- In 2d, the RG flow in the M-replicated O(n) models and Potts models with the bond-bond interaction can be explored with the (1-n)-expansion and (q-2)-expansion, respectively. This is true both for $M \to 0$ (non-unitary) and $M = 3, 4, 5, \ldots$ (unitary).

- For $M \to 0$, the symmetry enhancement at the non-perturbative fixed point (∃one-parameter extension of the Nishimori universality class at $n = 1$ or $q = 2$) is outstanding and suitable for a supergroup bootstrap of the spectrum with $C_{\text{eff}} = \left.\frac{\partial}{\partial M}\right|_{M=0} c_M \approx 0.4612(4)$.

- S-matrix method can yield exact results (e.g. $n_c = \sqrt{2} - 1$ for $M = 0$).

- CFTs for S_M-symmetry extended q-state Potts model. (Dotsenko-Jacobsen-Lewis-Picco 99∼)

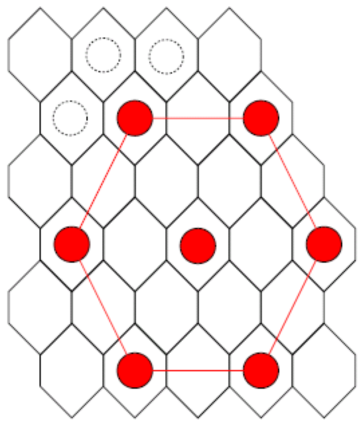
  Spectrum has been studied in (q-2)-expansion.
  Detailed numerical results are available, in particular, for $M = 3$.
  For large M, the interlayer coupling becomes weak ($1/M$-expansion). In the $M \to +\infty$ limit, $\Delta c_{FT} = \frac{-1}{125} = -0.008$ at $q = 3$ ($\Delta c_{FT} = \frac{-1}{8}$ at $q = 4$) is infinitesimally small compared with $M c_q$.

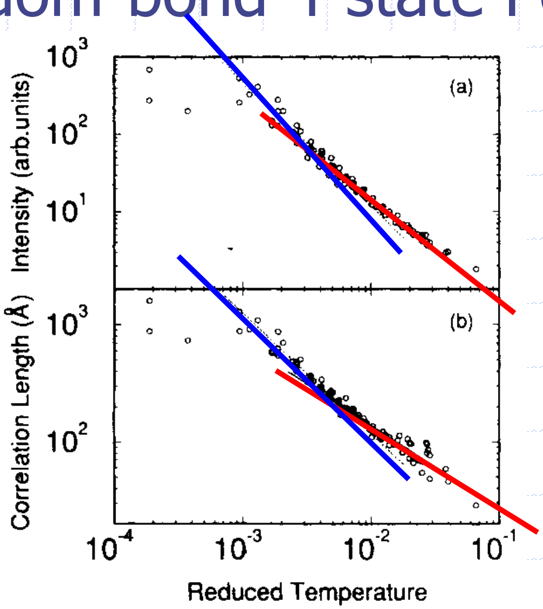
Backup Slides

Appendix:

Phase Transition in a Random System (Experiment)

- Adsorption of the Hydrogen atoms on the Ni(111) surface; order-disorder transition
- Some of sites are occupied by the Oxygen atoms.
- The universality of the Random-bond 4-state Potts model



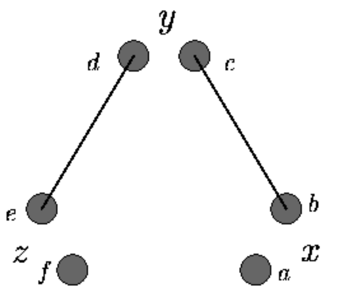


LEED (Low Energy Electron Diffraction)

L. Schwenger, K. Budde, C. Voges and H. Pfnür, *Phys. Rev. Lett* **73**, 296 (1994).

	Experiment		Theory	
	Pure	Impure	Ising	Four-state Potts
β	0.11 ± 0.01	0.135 ± 0.01	0.125	0.083
γ	1.2 ± 0.1	1.68 ± 0.15	1.75	1.167
ν	0.68 ± 0.05	1.03 ± 0.08	1.0	0.667

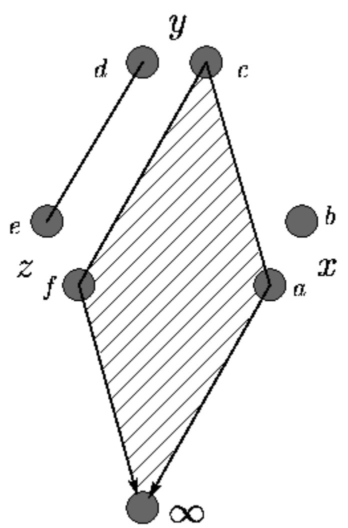
Example: two-loop β-function



$$A_{3,1}(r,\epsilon) = 4(M-2)(M-3)\int_{|y-x|,|z-x|<r} \langle \mathcal{E}(x)\mathcal{E}(y)\rangle_0 \langle \mathcal{E}(y)\mathcal{E}(z)\rangle_0 d^2z d^2y$$

$$= 8\pi(M-2)(M-3)\int_{y<r}\left(\frac{dy}{y^{1+16\epsilon}}\right)\int |z|^{-2+8\epsilon}|z-1|^{-2+8\epsilon}d^2z$$

$$= 16\pi^2(M-2)(M-3)\left(\frac{r^{8\epsilon}}{64\epsilon^2}\right).$$



$$A_{3,2}(r,\tilde{\epsilon}) = 4(M-2)\int_{|y-x|,|z-x|<r} \langle \mathcal{E}(x)\mathcal{E}(y)\mathcal{E}(z)\mathcal{E}(\infty)\rangle_0 \langle \mathcal{E}(y)\mathcal{E}(z)\rangle_0 d^2y d^2z$$

$$= 4(M-2)\mathcal{N}\int_{|y-x|,|z-x|<r} \langle V_{\overline{13}}(0)V_{13}(y)V_{13}(z)V_{13}(\infty)V_{\alpha_-}(u)V_{\alpha_-}(v)\rangle_0$$

$$\times \langle V_{13}(y)V_{13}(z)\rangle_0 d^2y d^2z d^2u d^2v$$

$$= 8\pi(M-2)\mathcal{N}\int_{y<r}\left(\frac{dy}{y^{1+16\epsilon}}\right)\times$$

$$\int |z|^{-4\alpha_{\overline{13}}\alpha_{13}}|z-1|^{+4\alpha_{13}^2-4\alpha_{\overline{13}}\alpha_{13}}|u-v|^{4\alpha_-^2}$$

$$|u|^{4\alpha_{\overline{13}}\alpha_-}|u-1|^{4\alpha_{13}\alpha_-}|u-z|^{4\alpha_{13}\alpha_-}$$

$$|v|^{4\alpha_{\overline{13}}\alpha_-}|v-1|^{4\alpha_{13}\alpha_-}|v-z|^{4\alpha_{13}\alpha_-}d^2z d^2u d^2v$$

Six-fold multiple integrals I

Generalized Complex Selberg Integral…Scattering Amplitude

(decomposing one complex three-fold integrals into two real ones $\{J_i^+, J_k^-\}$)

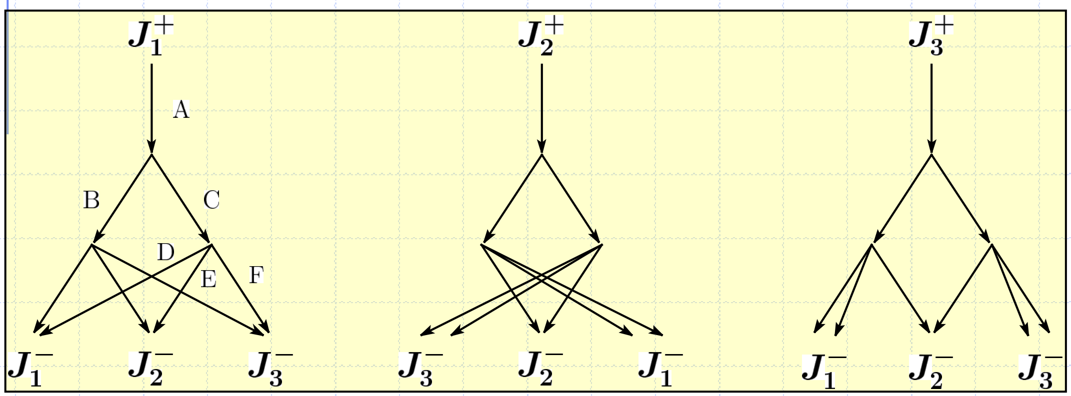
$$I = \int |z|^{2a}|z-1|^{2b}|u|^{2a'}|u-1|^{2b'}|v|^{2a'}|v-1|^{2b'}|z-u|^{2f}|z-v|^{2f}|u-v|^{2g} d^2z d^2u d^2v$$

$$\mathcal{M} = \begin{pmatrix} s(b)s(b')[s(b')+s(g+b')] & s(b+f)s(b')[s(b')+s(g+b')] & s(b+2f)s(b')[s(b')+s(g+b')] \\ s(b)s(b')[s(f+b')+s(f+g+b')] & s(b)s(b')^2+s(b+f)s(f+g+b') & s(b+f)s(b')[s(b')+s(g+b')] \\ s(b)s(f+b')[s(f+b')+s(f+g+b')] & s(b')s(f+b')+s(b)s(b')s(f+g+b') & s(b)s(b')[s(b')+s(g+b')] \end{pmatrix},$$

where $s(x) \equiv \sin \pi x$ is used.

$$J_1^+ = \int_0^1 du \int_0^u dv \int_0^v dz$$
$$J_2^+ = \int_0^1 du \int_0^u dz \int_0^z dv$$
$$J_3^+ = \int_0^1 dz \int_0^z du \int_0^u dv,$$

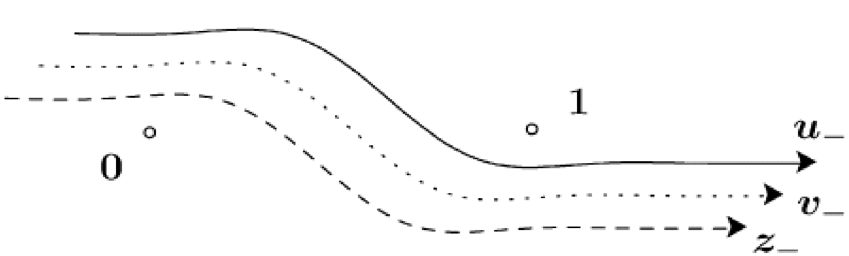
$$J_1^- = \int_1^\infty dz \int_z^\infty dv \int_v^\infty du$$
$$J_2^- = \int_1^\infty dv \int_v^\infty dz \int_z^\infty du$$
$$J_3^- = \int_1^\infty dv \int_v^\infty du \int_u^\infty dz,$$



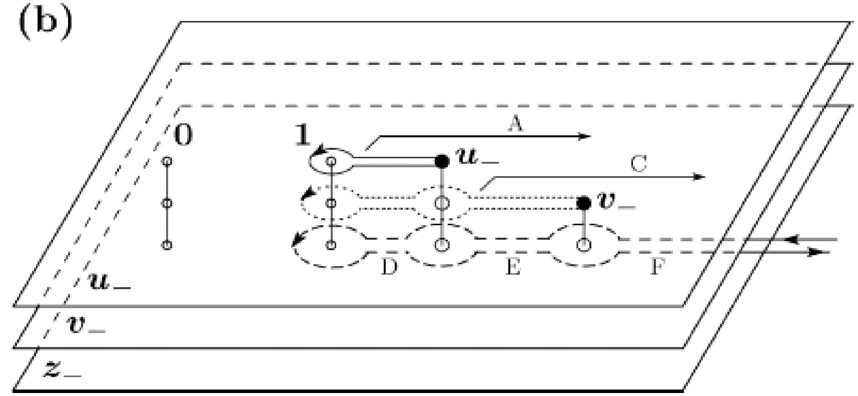
Interchanging adjacent variables on the real axis induces a pair of phase factors, and hence shifts arguments in sin-functions.

$$I = \sum_{i,k} J_i^+ \underbrace{s(*)s(*)s(*)}_{\mathcal{M}} J_k^-$$

(a)

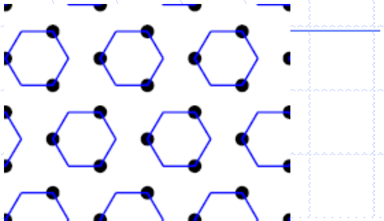


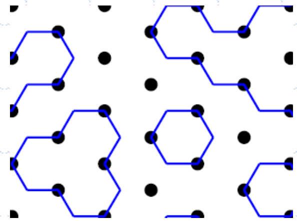
(b)



RG landscape for the disordered O(n) model $n = 50$

The zero temperature FP $\mathbf{Z_1}$

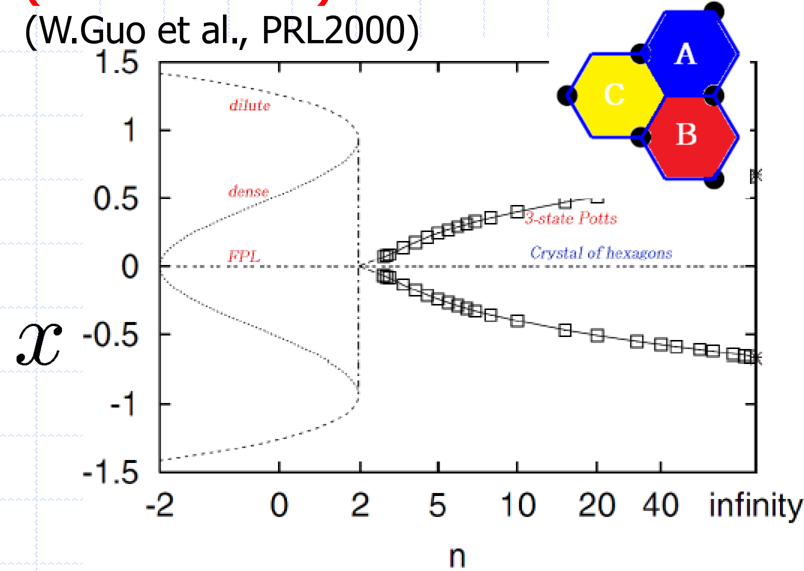


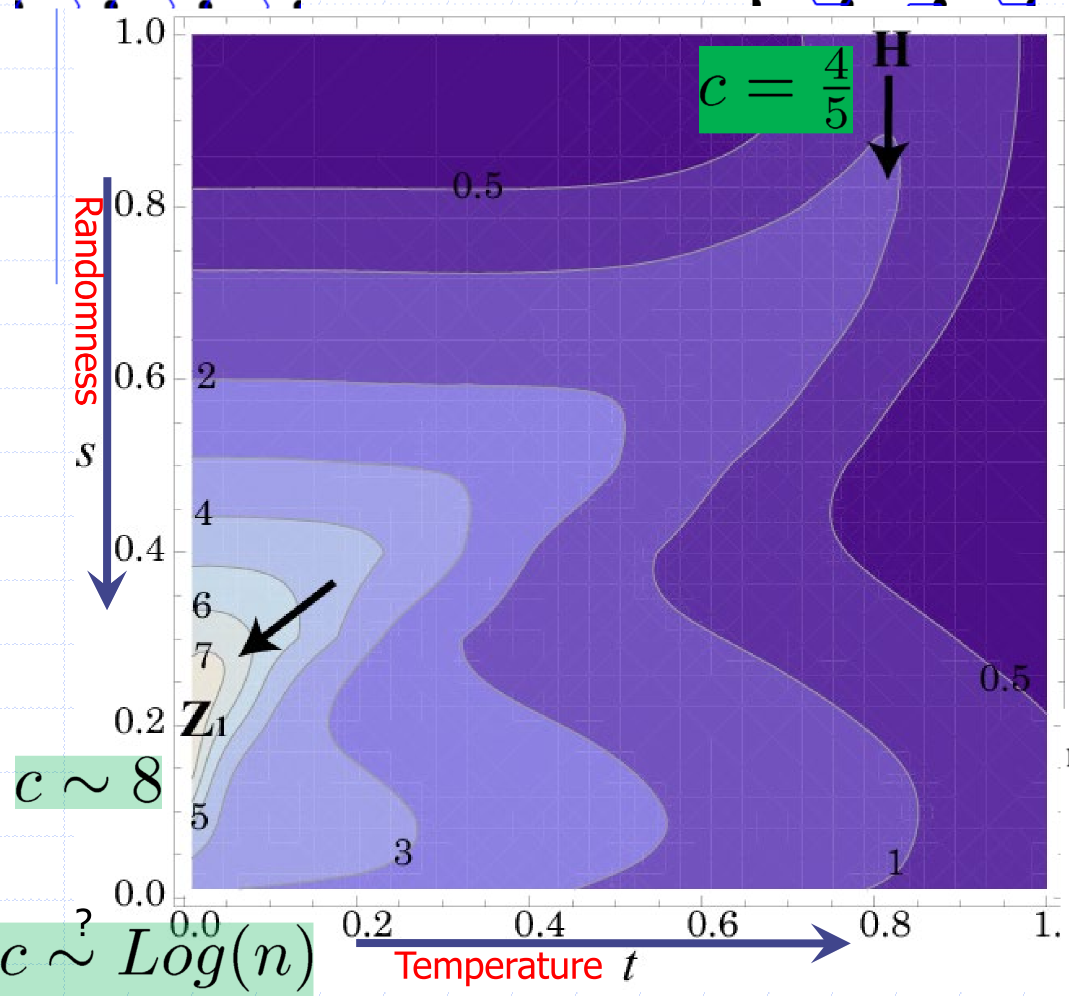


$$\mathrm{Tr}_{s_i} \prod_{\langle i,j \rangle} (1 + x s_i \cdot s_j)$$
$$\sim \mathrm{Tr}_{s_i} \prod_{\langle i,j \rangle} \exp(\beta s_i \cdot s_j)$$

Loop model and **Spin model** are in the same universality only when |n|<2.

n>2 loop model has a hard-hexagon (=3-state Potts) transition!
(W.Guo et al., PRL2000)





Random-bonds amount to random fields for 3-state Potts degrees of freedom.