

# RANDOM FIELD THEORY & PARISI-SOURLAS SUSY

BOOTSTAT 2021

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Slava Rychkov

[1912.01617]  
[2009.10087]  
[To appear]

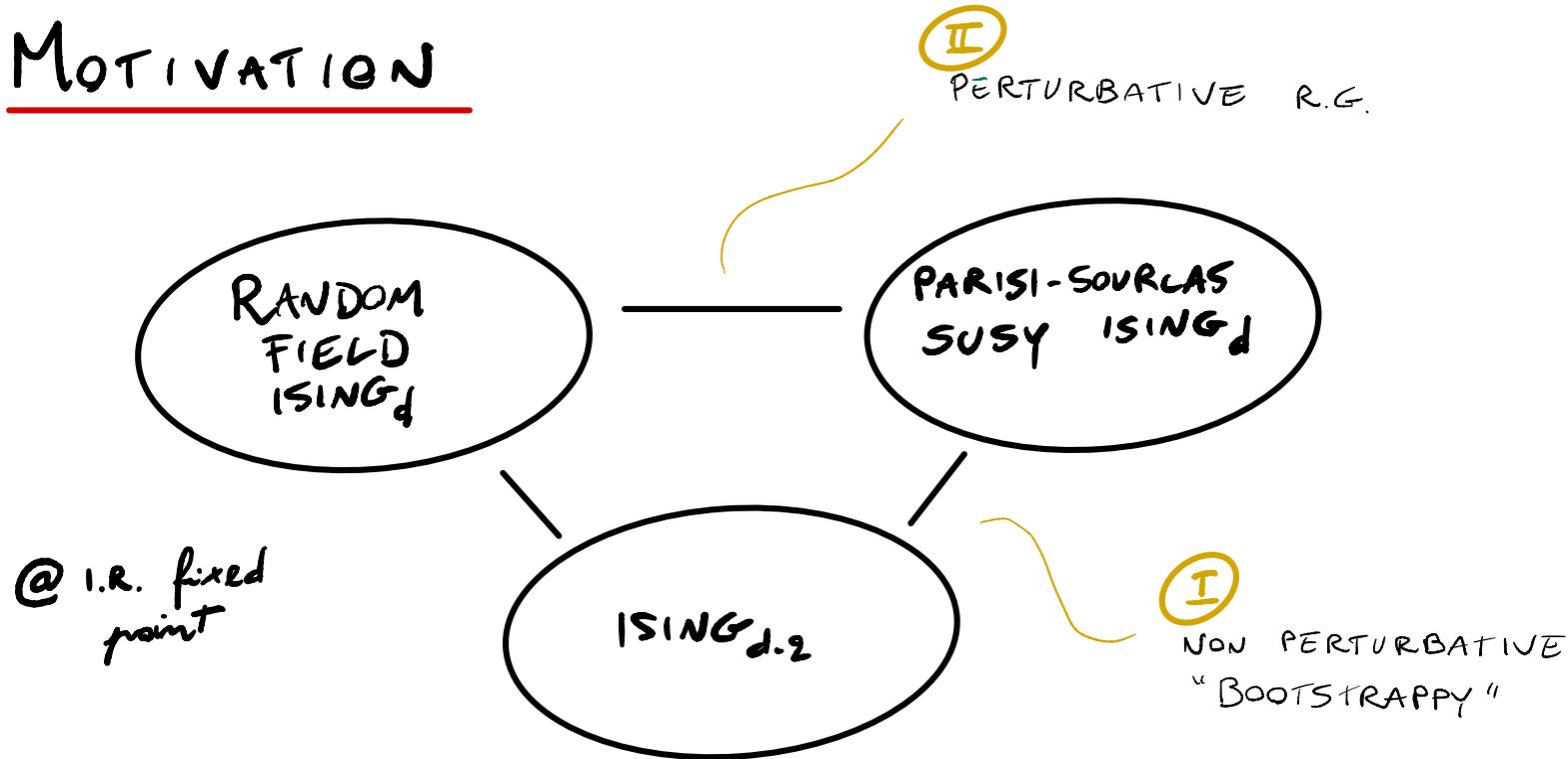


We are standing on The shoulders of giants

about 50 years on the subject

Imray, Ma, Aharony, Parisi, Sourlas, Young, Cardy, Harris,  
Imbraie, Brydges, Bricmont, Kupiainen, Wiese, Klein, Perez,  
Tissier, Targus, Brézin, De Dominicis, Orland, Tenenbaum  
Picco, Fytas, Martin-Mayer, Bolog, Baczys,  
Franz, Ricci-Tersenghi, Hikami, Villain, Adler  
Lancaster, Marinari, Angelini, Lucibello, Gafner,  
.... many more

# MOTIVATION

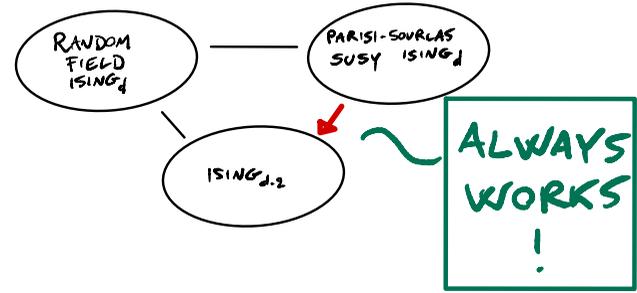


- o understand when/why works
- o standard QFT methods (e.g.  $\epsilon$ -expansion)

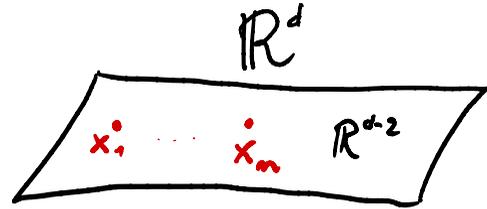
[Talk by M. PICCO]

$\left\{ \begin{array}{l} d=3,4 \text{ fails} \\ d=5 \text{ works (?) } \\ d=6 \text{ upper critical} \end{array} \right.$

# I DIMENS. REDUCTION



- define PARISI-SOURLAS CFT
- understand dimensional reduction to CFT<sub>d-2</sub>

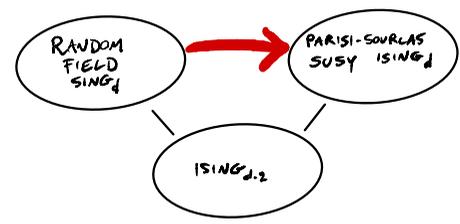


DECOUPLING  
∞-many  
OPERATORS

OPERATORS  
 $T_{M\bar{V}}$   
 OPE  
 CONF. BLOCKS  
 $g_{\Delta, \ell}^{d-2} = g_{\Delta, \ell}^d$  SUPER C.B.

$$g_{\Delta, \ell}^{d-2} = \sum_{\text{FINITE}} g_{\Delta+m, \ell+m}^d$$

# II The emergence of SUSY



RANDOM FIELD



REPLICA TRICK



CARDY Transform

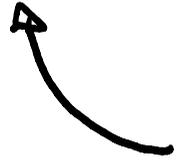
[ '84 CARDY ]  
+  
[ II ]

I.R. LEADERS THEORY

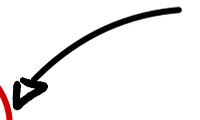
SUSY



(-2) FREE BOSONS  
↓  
(2) FREE FERMIONS



~~SUSY~~



RANDOM  
FIELD

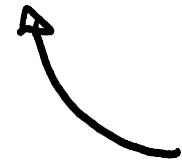


REPLICIA  
TRICK

SUSY



(-2) FREE BOSONS  
↓  
(2) FREE FERMIONS



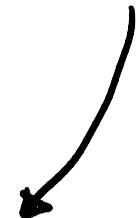
~~SUSY~~



I.R.  
LEADERS  
THEORY



CARDY  
Transform



# RANDOM FIELD THEORY

R.F.I.M.

$$V = \lambda \int \phi^4$$

$$S^h[\phi] = \int d^d x \frac{1}{2} (\partial\phi)^2 + V(\phi) + h\phi$$

- NOT A QFT

$$\int \mathcal{D}h \underbrace{P(h)}_{\text{GAUSSIAN}} h(x) h(y) = H \delta(x-y)$$

GAUSSIAN

# REPLICA THEORY

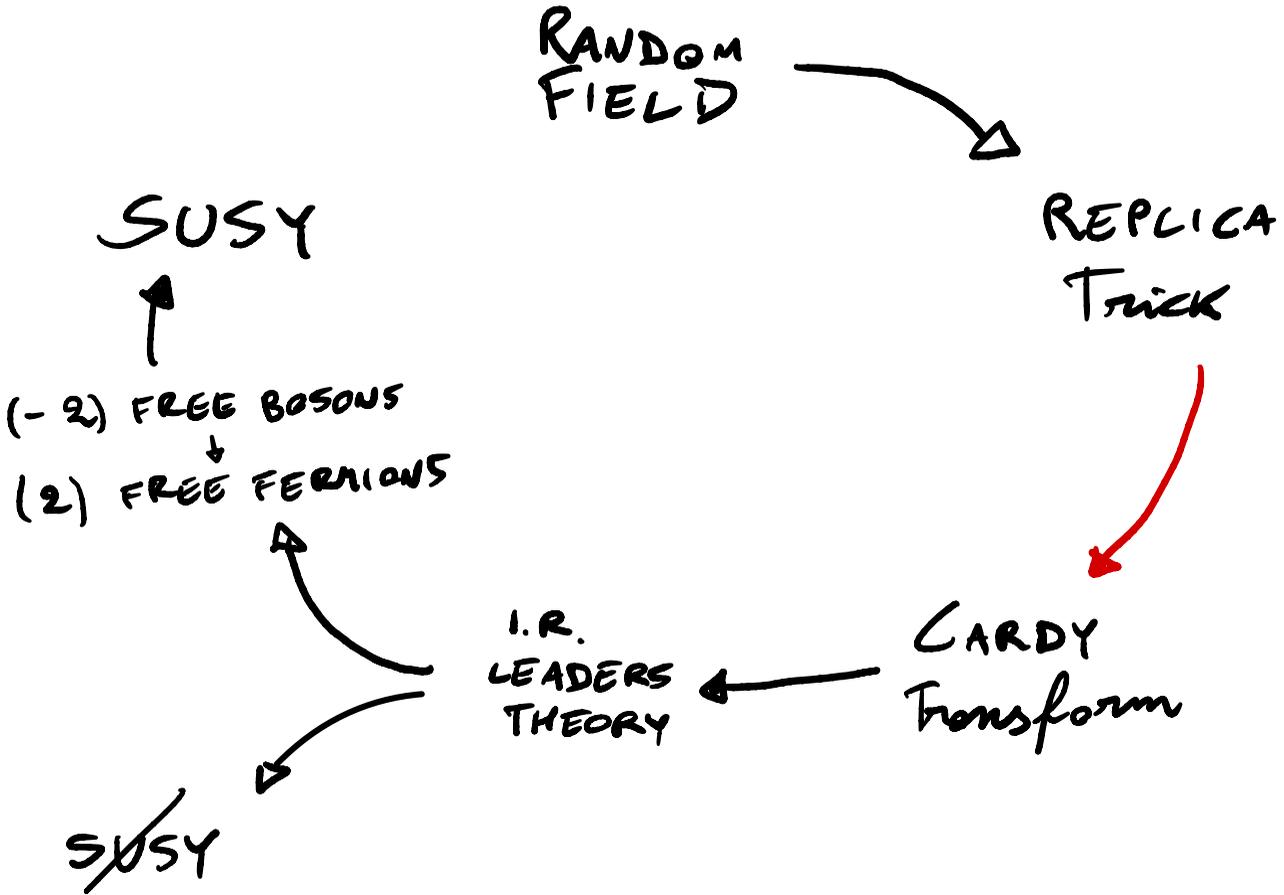
$$S_m[\phi^a] = \int d^d x \frac{1}{2} \sum_{i=1}^m (\partial\phi_i)^2 + \sum_{i=1}^m V(\phi_i) - \frac{H}{2} \left( \sum_{i=1}^m \phi_i \right)^2$$

- QFT
- $m \rightarrow 0$  COUPLED REPLICAS

# REPLICA TRICK

$$\begin{aligned}
 \overline{\langle \mathcal{O}(\phi) \dots \rangle_{R.F.}} &\equiv \int \mathcal{D}h P(h) \frac{1}{Z_h} \int \mathcal{D}\phi \mathcal{O}(\phi) \dots e^{-S_h[\phi]} = \\
 &= \int \mathcal{D}h P(h) \frac{1}{Z_h} \frac{Z_h^{n-1}}{Z_h^{n-1}} \int \mathcal{D}\phi \mathcal{O}(\phi) \dots e^{-S^h[\phi]} = \\
 &= \int \mathcal{D}h P(h) \left( \frac{1}{Z_h^n} \right) \int \mathcal{D}\phi_i \mathcal{O}(\phi_i) \dots e^{-\sum_{i=1}^n S^h[\phi_i]} \\
 &= \lim_{n \rightarrow 0} \int \mathcal{D}\phi_i \mathcal{O}(\phi_i) \dots e^{-S_n[\phi_i]} = \lim_{n \rightarrow 0} \langle \mathcal{O}(\phi_i) \dots \rangle_{S_n}
 \end{aligned}$$

$$S_n \equiv \int d^d x \sum_{i=1}^n (\partial \phi_i)^2 + \sum_{i=1}^n V(\phi_i) - H \left( \sum_{i=1}^n \phi_i \right)^2$$



# REPLICA PROPAGATOR

$$\mathcal{L}_m^{\text{FREE}} = \frac{1}{2} \sum (\partial \phi_i)^2 - H (\sum \phi_i)^2 \quad (V=0)$$

$$\langle \phi_i(-k) \phi_j(k) \rangle = \frac{\delta_{ij}}{k^2} + \frac{H \pi_{ij}}{|k|^4 - \cancel{mH} k^2}$$

$m=0$

→  $\Delta \phi = ?$

$\phi$  is not a scaling field → R.G. in  $\phi_i$   
COMPLICATED

---

Can we decouple the powers of  $|k|$  ?

CARDY TRANSFORM

# CARDY TRANSFORM

$$\begin{cases} \varphi = \frac{1}{2} (\phi_1 + \rho) \\ w = \phi_1 - \rho \\ \chi_i = \phi_i - \rho \end{cases}$$

$$\rho \equiv \frac{\phi_2 + \dots + \phi_m}{m-1}$$

$$i=2, \dots, m$$

$$\sum_{i=2}^m \chi_i = 0$$

$$\rightarrow \mathcal{L}_{\text{CARDY}}^{\text{FREE}} = \partial \varphi \partial w - H w^2 + \frac{1}{2} (\partial \chi_i)^2 \quad (m=0)$$

$$\langle \varphi(k) \varphi(-k) \rangle = \frac{H}{|k|^4}$$

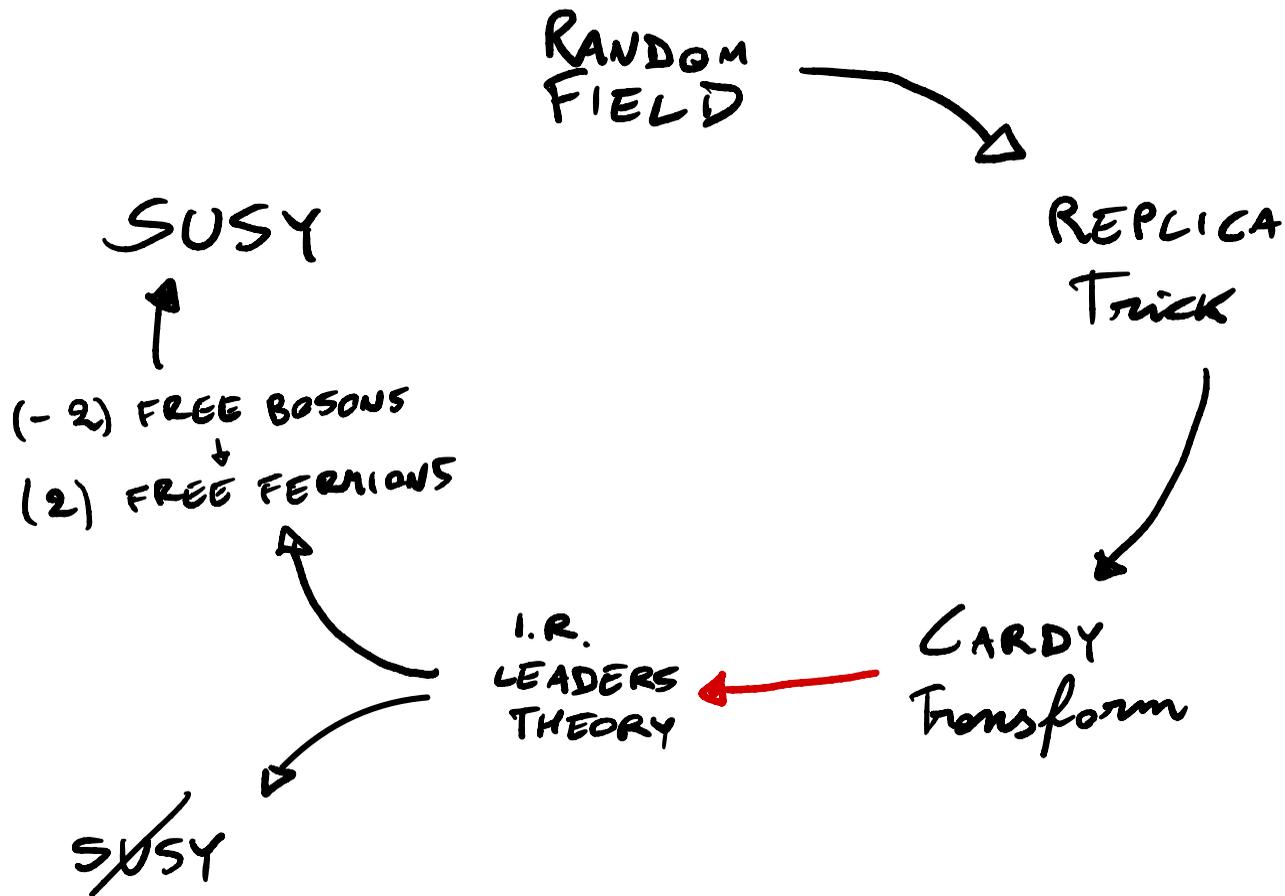
$$\langle \varphi(k) w(-k) \rangle = \frac{1}{k^2}$$

$$\langle \chi_i(k) \chi_j(-k) \rangle = \frac{1}{k^2} \left( \delta_{ij} - \frac{\pi_{ij}}{m-1} \right)$$

$$\rightarrow \Delta \varphi = \frac{d}{2} - 2$$

$$\Delta \chi_i = \frac{d}{2} - 1$$

$$\Delta w = \frac{d}{2}$$



# REPLICA R.G. in Cardy's variables

## & LEADERS THEORY

- ① Write all  $S_n$  singlets (+  $Z_2$  for R.F. ISING) earlier in terms of  $\phi_i$ . [98 BRÉZIN, DE DOMINICIS]

$$G_K \equiv \sum \phi_i^K$$

$$G_{K(\mu\nu)} \equiv \sum \phi_i^{K-1} \partial_\mu \partial_\nu \phi_i$$

$$G_{K(\rho)(\nu)} \equiv \sum \phi_i^{K-2} \partial_\rho \phi_i \partial_\nu \phi_i$$

⋮

② Map  $S_m$  singlets To Cardy basis ( $m=0$ )

$$\left. \begin{aligned} \mathcal{O}_{2(\mu)(\mu)} &= \partial\varphi \partial w + \frac{1}{2} (\partial X_i)^2 \\ \mathcal{O}_1^2 &= w^2 \end{aligned} \right\} \begin{array}{l} \mathcal{L}^{\text{FREE}} \\ \mathcal{L}^{\text{CARDY}} \end{array}$$

$$\mathcal{O}_2 = 2\varphi w + X_i^2 \quad \sim \text{mass term} \quad \text{strongly relevant}$$

$$\mathcal{O}_4 = \underbrace{(4w\varphi^3 + 6X_i^2\varphi^2)}_{(\mathcal{O}_4)_L} + \underbrace{4\varphi X_i^3}_{(\mathcal{O}_4)_{F_1}} + \underbrace{(X_i^4 - 6\varphi w X_i^2)}_{(\mathcal{O}_4)_{F_2}}$$

$$- \underbrace{2wX_i^3}_{(\mathcal{O}_4)_{F_3}} + \underbrace{\left(\frac{3}{2}w^2 X_i^2 + \varphi w^3\right)}_{(\mathcal{O}_4)_{F_4}}$$

$\mathcal{O}_L$  leader

$\mathcal{O}_{F_i}$  followers

### ③ Kill The FOLLOWERS! (in The I.R.)

R.G. step

$$g(U_L + U_{F_1} + \dots) \xrightarrow[\substack{\text{INTEGRATING} \\ \text{OUT } [\frac{1}{b}, 1] \\ b > 1}]{\text{PRESERVED BY } S_m} \tilde{g}(U_L + U_{F_1} + \dots)$$

$$\tilde{g}(U_L + U_{F_1} + \dots) \xrightarrow{\text{RESCALING}} g(b) \left( U_L + \frac{U_{F_1}}{b} + \frac{U_{F_2}}{b^2} + \dots \right)$$

↓ I.R. ↓  
○ ○

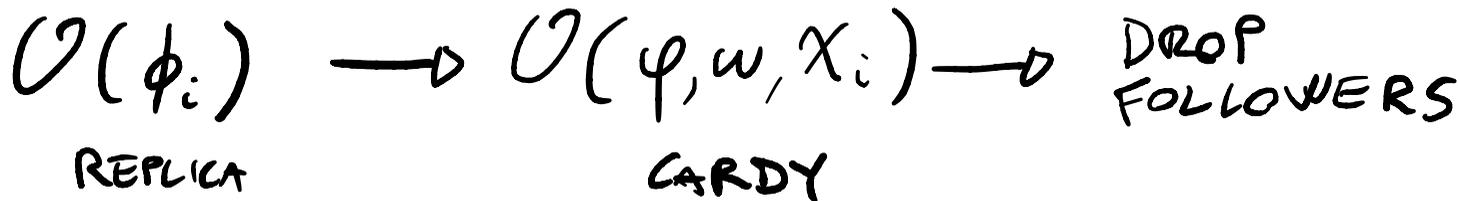
# SUMMARY

LEADERS  
THEORY

$$= \left\{ \mathcal{O}_L : \mathcal{O} \text{ is } S_n\text{-singlet} \right. \\ \left. \mathbb{Z}_2\text{-singlet} \right\}$$

ISING

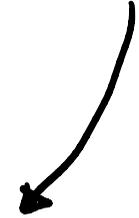
EASY ALGORITHM TO BUILD  $\mathcal{O}_L$ :



RANDOM  
FIELD



REPLICIA  
TRICK



CARDY  
Transform



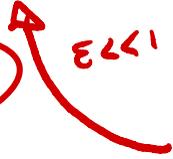
I.R.  
LEADERS  
THEORY

SUSY



(-2) FREE BOSONS  
↓  
(2) FREE FERMIONS

$d = 6 - \epsilon$



~~SUSY~~

R.G. in  $d = 6 - \epsilon$

( $\epsilon \ll 1$ )

$$\left. \begin{aligned} \sigma_{2(\mu)(\mu)} &= \partial\varphi \partial w + \frac{1}{2} (\partial X_i)^2 \\ \sigma_1^2 &= w^2 \end{aligned} \right\} \begin{array}{l} \text{FREE} \\ \text{LADY} \end{array}$$

$$\sigma_2 = 2\varphi w + X_i^2 \sim \text{strongly relevant} \quad \text{need Tuning}$$

$$(\sigma_4)_L = (4w\varphi^3 + 6X_i^2\varphi^2) \sim \begin{array}{l} \text{weakly} \\ \text{relevant} \\ \Delta = 6 - 2\epsilon \end{array} \quad \text{perturbative R.G.}$$

All other leaders  $\sim$  IRRELEVANT ( $\epsilon \ll 1$ )

$$\mathcal{L} = \left[ \partial\varphi \partial w + \frac{1}{2} (\partial X_i)^2 - \frac{H}{2} w^2 \right] + \lambda (4w\varphi^3 + 6 X_i^2 \varphi^2)$$

$$\mathcal{L} = \left[ \partial\varphi\partial\omega + \frac{1}{2} (\partial\underline{\chi}_i)^2 - \frac{H}{2}\omega^2 \right] + \lambda (4\omega\varphi^3 + 6\underline{\chi}_i^2\varphi^2)$$

○ QUADRATIC in  $\chi_i$

○  $\frac{1}{2} \sum_i \chi_i (-\partial^2 + A(\varphi)) \chi_i \longleftrightarrow \psi (-\partial^2 + A(\varphi)) \bar{\psi}$  *Grassmann numbers*

$$\left[ \det(-\partial^2 + A) \right]^{-\frac{n-2}{2}} \longleftrightarrow \left[ \det(-\partial^2 + A) \right]$$

○  $O(-2) \longleftrightarrow SP(2)$

$$\mathcal{L}_{\text{SUSY}} = \left[ \partial\varphi\partial\omega + \partial\psi\partial\bar{\psi} - \frac{H}{2}\psi\bar{\psi} \right] + \lambda (4\omega\varphi^3 + 6\varphi^2\psi\bar{\psi})$$

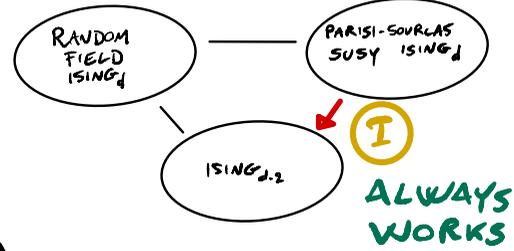
→ SUSY LAGRANGIAN

# SUPERSPACE ACTION

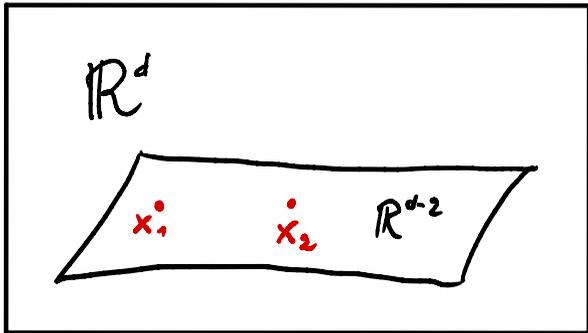
$$S = \int d^d x d\theta d\bar{\theta} \left[ -\frac{1}{2} \Phi D^2 \Phi + \lambda \Phi^4 \right]$$

$$\left\{ \begin{array}{l} \Phi = \varphi + \theta \bar{\varphi} + \bar{\theta} \psi + \theta \bar{\theta} w \\ D^2 = \partial^2 - 4 \partial_{\theta} \partial_{\bar{\theta}} \end{array} \right.$$

# DIMENSIONAL REDUCTION



$$\text{e.g. } \langle \varphi(x_1) \varphi(x_2) \rangle_{\text{SUSY}} = \langle \hat{\phi}(x_1) \hat{\phi}(x_2) \rangle_{d-2}$$



$$\mathcal{S}_{d-2} \equiv \int dx^{d-2} (\partial \hat{\phi})^2 + \lambda \hat{\phi}^4$$

WILSON FISHER

Very well known!

$$(d-2) = 4 - \epsilon$$

RANDOM  
FIELD

REPLICA  
TRICK

SUSY

(-2) FREE BOSONS

(2) FREE FERMIONS

I.R.  
LEADERS  
THEORY

CARDY  
Transform

$d = 6 - \epsilon$

$\epsilon \sim 1$

OTHER RELEVANT ?  
PERTURBATIONS ?

# TYPES OF LEADERS

- SUSY - WRITABLE  $O(m-2)$

$$\frac{1}{2} \chi_i^2 f(\varphi, \omega) \longrightarrow \psi \bar{\psi} f(\varphi, \omega)$$

- SUSY - NULL

$$(\chi_i^2)(\chi_j^2) \longrightarrow (\psi \bar{\psi})^2 = 0$$

- NON - SUSY - WRITABLE  $S_{m-2}$   
 $\chi_i^{K \geq 2}$

# CLASSIFICATION of The leaders (up to $\Delta \leq 12$ ) @ $d=6$

- a lot of SUSY-WRITABLE leaders  
→ mapped to W.F.  
→ known IRRELEVANT (besides  $\hat{\phi}^2$ )

- a few { NON-SUSY-WRITABLE  
SUSY-NULL

$$\hookrightarrow F_K = \sum_{i,j=1}^n (\phi_i - \phi_j)^K \quad = 4, 6, 8, \dots$$

Feldman  
[2002]

$F_4, F_6$  lowest dimensional

# COMPUTATION of ANOMALOUS DIMENSIONS

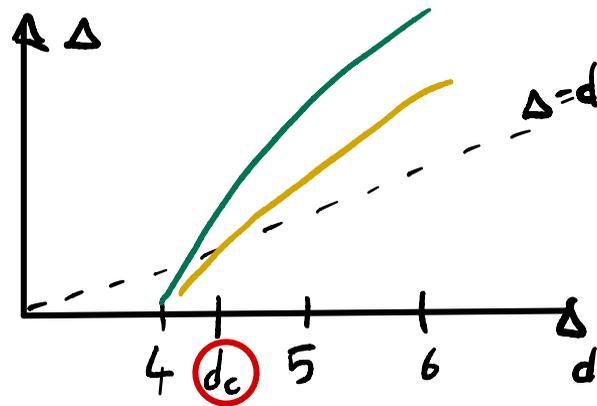
$$\text{Feldman} : F_K = \sum_{i,j=1}^n (\phi_i - \phi_j)^K$$

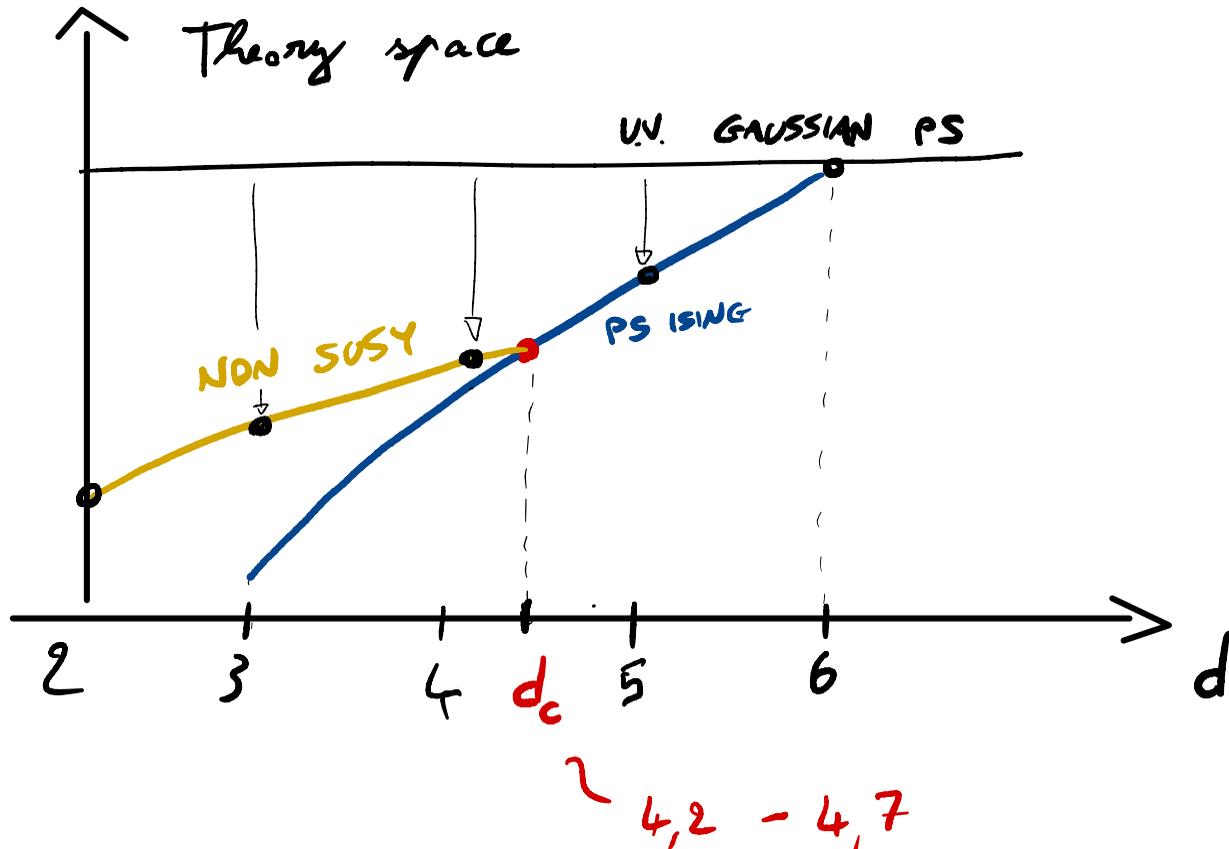
1st SUSY-NULL  $(F_4)_L = (\chi_i^2)^2$

1st NON-SUSY-WRITABLE  $(F_6)_L = (\chi_i^3)^2 - \frac{3}{2} (\chi_i^2)(\chi_i^4)$

$$\Delta_{(F_4)_L} = 8 - 2\varepsilon - \frac{8}{27} \varepsilon^2$$

$$\Delta_{(F_6)_L} = 12 - 3\varepsilon - \frac{7}{9} \varepsilon^2$$



$\phi^4$  $\mathbb{Z}_2$ -sym

# SUMMARY & FUTURE DIRECTIONS

- Defined PARISI-SOURLAS CFT
  - Understood dimensional reduction P.S. CFT
  - Conformal block relation
- } (I)
- Defined LEADER THEORY for R.G.
  - classified LEADERS
  - found perturbations that cross marginality
- $$(F_4)_L = (X_i^2)^2 \quad (F_6)_L = (X_i^3)^2 - \frac{3}{2} (X_i^2)(X_i^4)$$
- $\exists$  SUSY in  $d_c = 4, 2-4, 7 < d < 6$
- } (II)

- 
- DIMENSIONAL REDUCTION  $4d-2d$
  - BETTER UNDERSTAND LEADER THEORY  $\rightarrow$  BOOTSTRAP?
  - Applications to other models (e.g.  $V = \phi^3$ )