

Conformal Field Theories near an Edge

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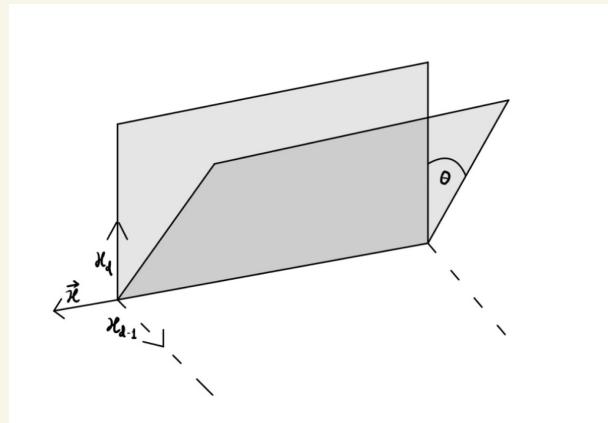
Main References:

"Critical Behaviors at an Edge" J.L. Cardy
J. Phys A 16 3617 1983

"Conformal Bootstrap near the Edge" AA
arXiv 2103.03132

$$\mathcal{H} = (\vec{x}, x_{d-1}, x_d)$$

Setup:



∞ -planes
 Θ -parameter
Critical
Exponents?

- 1 d-dimensional bulk (Wedge) ✓
- 2 (d-1)-dimensional boundaries S
- 3 (d-2)-dimensional Edge E

Lagrangian/RG approach (Cardy 83')

Landau - Ginzburg $O(N)$ model

$$F[\phi] = \int_V d^d x \left[\frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{1}{2} m_0^2 (\phi_i \phi^i) + \frac{1}{4} \mu_0 (\phi_i \phi^i)^2 \right]$$
$$+ \int_S d^{d-1} x \left(\frac{1}{2} C(\phi^i \phi_i) \right)$$

$C > 0$ (Ordinary transition $T_c^{(v)} = T_c^{(s)}$)

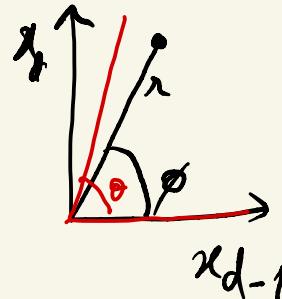
RG : $C \rightarrow \infty \rightarrow$ Dirichlet Boundary Condition
 $\phi^i = 0 @ S$
↳ Good for Perturbation theory

Mean Field Theory 2- PT Function $G_0(x, x')$

$$G_0(x, x') = \int \frac{d^{d-2} \vec{k}}{(2\pi)^{d-2}} e^{i \vec{k} \cdot (\vec{x} - \vec{x}')} J_0(\vec{k}, r, \phi, r', \phi')$$

w/ $x_{d-1} = r \cos \phi$

$x_d = r \sin \phi$



- $(-\nabla^2 + m_0^2) G_0(x, x') = \delta^{(d)}(x - x')$

w/ DBC at 2S

$$J_0(\vec{k}, r, \phi, r', \phi') =$$

$$= \frac{2}{\theta} \sum_{m=1}^{\infty} \int_0^{\infty} \mu d\mu \frac{J_{\frac{m\pi}{\theta}}(\mu r) J_{\frac{m\pi}{\theta}}(\mu r')}{\underbrace{(\vec{k}^2 + m^2) + \mu^2}_{k^2}} \sin\left(\frac{m\pi}{\theta}\phi\right) \sin\left(\frac{m\pi}{\theta}\phi'\right)$$

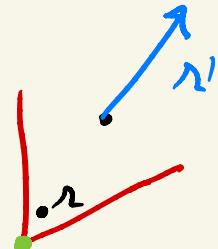
At Critical Point $m^2 = \frac{1}{\epsilon_2} \rightarrow 0$

$$G_0 \sim \frac{1}{|\vec{x} - \vec{x}'|^{d-2 + m_{\phi,0}}}$$



Asymptotic limits:

i) $\vec{x} = \vec{x}', r$ fixed, $r' \rightarrow \infty$

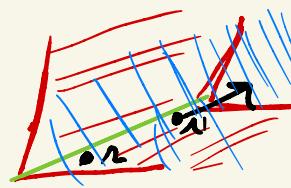


Bulk-Edge ($m=1$ dominates)

$$G_0 \sim A(\theta) \frac{\lambda^{\frac{\pi}{\theta}}}{\lambda'^{d-2 + \frac{\pi}{\theta}}} \gamma \sin\left(\frac{\pi}{\theta}\phi\right) \sin\left(\frac{\pi}{\theta}\phi'\right)$$

ii) r, r' fixed $|\vec{x} - \vec{x}'| \rightarrow \infty$

$$G_0 \sim \frac{(rr')^{\frac{\pi}{\theta}}}{|\vec{x} - \vec{x}'|^{d-2 + \frac{2\pi}{\theta}}} \gamma_{E,E}$$



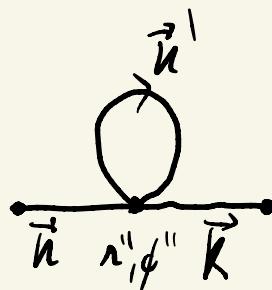
With a magnetic field, can also derive:

Edge Magnetization ρ_E ; Edge Susceptibility χ_E

One loop calculation ($4 - \varepsilon$)

$$\gamma(\vec{u}, r, \phi, r', \phi') = \gamma_0 + \gamma_1$$

$$= \text{---} \vec{k} \text{---} +$$



$$\begin{aligned} \gamma_1 &= -(N+2) M_0 \int_0^\infty r^4 dr' \int_0^\theta d\phi'' \gamma_0(\vec{u}, r, \phi, r'', \phi'') \gamma_0(\vec{u}, r', \phi', \\ &\quad \times \frac{\int^{d-2} \vec{u}'}{(2\pi)^{d-2}} \gamma_0(\vec{R}', r'', \phi'', r'', \phi'') \end{aligned}$$

when $r \rightarrow 0$, take $\sin(\frac{\pi}{8}\phi)$

Very hard Calculation: Extract by r

Appears only from $r'' \sim r \rightarrow 0$: Edge effect.

Assuming log exponents: $G \sim r^{1/2-d} \left(\frac{r}{r_*}\right)^\gamma_{V,E}$

$$\gamma_{V,E} = \frac{\pi}{\theta} - \epsilon \left(\frac{5 \frac{\pi^2}{\theta^2} + 1}{6 \frac{\pi}{\theta}} \right) \frac{(N+2)}{2(N+8)}$$

Agrees with $\gamma_{V,S}$ for $\theta = \pi$.

Renormalization of Edge Operators

New exponents \longleftrightarrow New edge UV divergences

Leading Edge mode: $A^{(1)}(\vec{x}) = \lim_{r \rightarrow 0} \frac{\varphi(\vec{x}, r, \phi)}{r^{\frac{1}{2}} \sin \frac{\pi}{\theta} \phi}$

$$H + h_2 \int_E d\vec{x} A^{(1)}(\vec{x}) \rightarrow Z_{h_2}$$

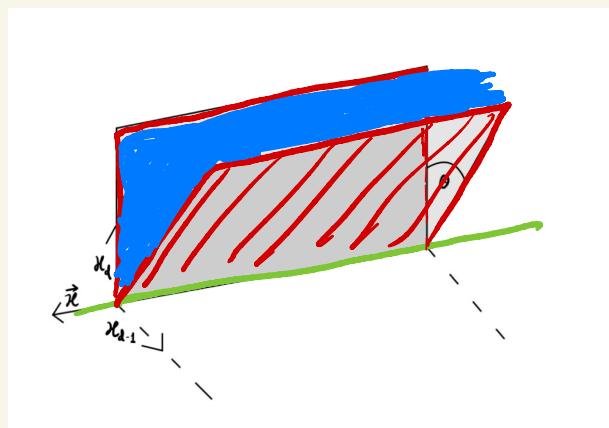
Edge Renormalization Constant

\rightarrow Bulk-Edge RG Equation

Recover 1-loop result

Bootstrap Approach

(AA '21)



Critical Bulk
(Homogeneous)

$SO(d+1, 1)$

↓
1 Boundary

$SO(d, 1)$

Breaks
1 Trans.
1 SCT
 $d-1$ Rot

↓
2 Boundary /
1 Edge

Breaks
1 Trans
1 SCT
 $d-2$ Rot

Conformal → $SO(d-1, 1)$
Symmetries of the
 $d-2$ dimensional Edge for $d \geq 3$.

The Boundary theories are themselves BCFTs.
Their Boundary is The Edge.

Kinematics Zoo

$O \hat{\theta} \hat{\theta}$

Bulk 1-pt Function

$$\langle O(\vec{x}, x_{d-1}, x_d) \rangle = \frac{g(\eta, \theta)}{(2x_{d-1})^{\Delta}} \rightarrow \begin{array}{l} \text{Cross-Ratio} \\ \text{Parameter} \end{array}$$

Bulk scaling dimension

$$\eta = \frac{x_{d-1}}{x_d} = \tan \phi \rightsquigarrow \text{cannot change by a conformal transformation}$$

Non-trivial position dependent 1-pt function.
 ((Can it be fixed by Yamada eq. ??))

Bulk-Edge 2-pt Function

$$\langle O_1(\vec{x}, x_{d-1}, x_d) \hat{O}_2(\vec{x}', 0, 0) \rangle = \frac{g(\phi, \theta)}{r^{2\hat{\Delta}_2} (2x_{d-1})^{\Delta_1 - \hat{\Delta}_2}}$$

Edge scaling dimension

Kinematics Zoo Continued

Boundary 1-pt function

$$\langle \hat{O}_1(\vec{x}, 0, x_d) \rangle = \frac{\hat{\alpha}^1(\theta)}{(2x_d)^{\Delta_1}} = \frac{\hat{\mu}_1^1(\theta)}{(2x_d)^{\Delta_1}}$$

Boundary 1-pt
Coefficient / Boundary
to Edge 1 Coefficient.

Boundary
Scaling
Dimension

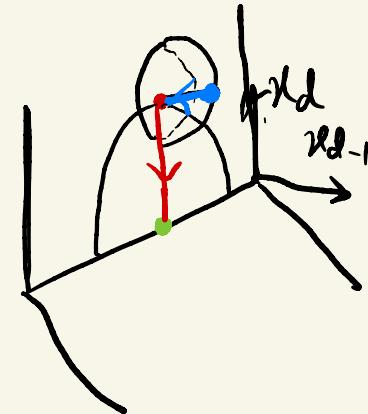
Boundary - Edge 2-pt function

$$\langle \hat{O}_1(\vec{x}, 0, x_d) \hat{O}_2(\vec{x}', 0, 0) \rangle = \frac{\hat{\mu}_2^1(\theta)}{(2x^2)^{\Delta_2} (2x_d)^{\Delta_1 - \Delta_2}}$$

Boundary to Edge
OPE coefficient.

Boundary OPE

We can still use boundary OPE
(twice!) (when it converges)



$$O(\vec{x}, x_{d-1}, x_d)$$

$$= \frac{a_0}{(2x_{d-1})^\Delta} + \sum_m \frac{\mu_m}{(2x_{d-1})^{\Delta - \Delta_m}} D[x_{d-1}, \rho_1, \rho_2] \hat{O}_m(\vec{x}, x_d)$$

Block Expansion (Bulk 1-pt)

$$\langle O(\vec{x}, x_{d-1}, x_d) \rangle_\theta = \frac{a_0(\theta)}{(2x_{d-1})^\Delta} + \sum_m \frac{\mu_m}{(2x_{d-1})^{\Delta - \hat{\Delta}_m}} D[\vec{x}, p] \hat{a}_m^{(p)} / (2x_d)^{\hat{\Delta}_m}$$

Known in BCFT

New Edge data

Using this we derive a block expansion:

$$\langle O(\vec{x}, x_{d-1}, x_d) \rangle_\theta = \frac{1}{(2x_{d-1})^\Delta} \left(a_0(\theta) + \sum_m C_m(\theta) f_{\text{wall}}(\hat{\Delta}_m, h) \right)$$

$$C_m(\theta) = \mu_m \hat{a}_m^{(p)}$$

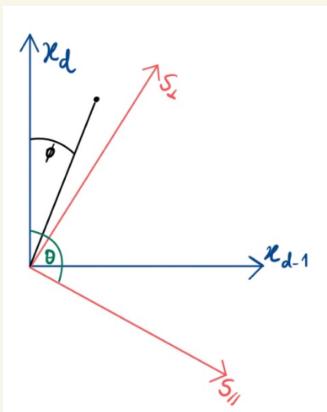
"Wall-channel" Conformal Blocks:

$$f_{\text{wall}}(\hat{\Delta}_m, h) = h^{\hat{\Delta}_m} {}_2F_1\left(\frac{\hat{\Delta}_m}{2}, \frac{\hat{\Delta}_m + 1}{2}; \frac{3}{2}; \frac{-d}{2} + \hat{\Delta}_m, -h^2\right)$$

Ramp Channel and Grossing Equation

We expanded the conductor as $x_{d-1} \rightarrow 0$ (wall channel)

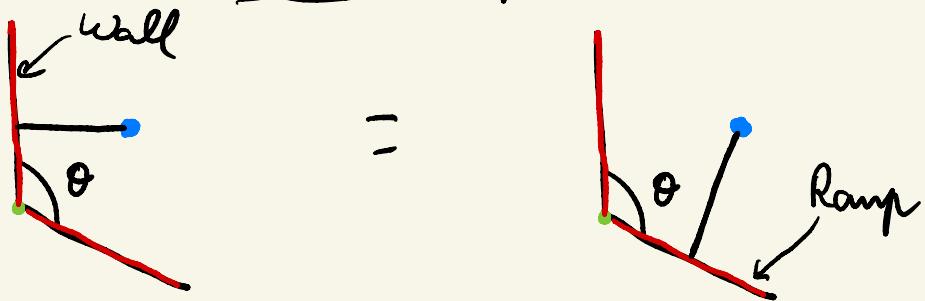
Can also expand on other **Boundary** (ramp channel).



Grossing to Ramp channel

$$\begin{cases} x_{d-1} \rightarrow s_\perp \\ x_d \rightarrow s_{\parallel\parallel} \\ \eta \rightarrow \zeta \equiv \tan(\theta - \phi) \end{cases}$$

Wall = Ramp



$$\frac{g_\theta(\phi)}{(2x_{d-1})^\Delta} = \frac{g'_\theta(\theta - \phi)}{(2s_\perp)^\Delta}$$

$$a_0 + \sum_m C_m(\theta) f_{\text{wall}}(\lambda_m, \eta) = \left(\frac{\sin \phi}{\sin(\theta - \phi)} \right)^\Delta \left(a'_0 + \sum_m C'_m(\theta) f_{\text{Ramp}}(\lambda_m, \zeta) \right)$$

Example Solutions

Simple Boundary

$$\langle O(\vec{x}, x_{d-1}, \vec{y}_d) \rangle = \frac{a_0}{(2x_{d-1})^\Delta}$$

$$\frac{a_0}{\eta^\Delta} = \sum_{m=0}^{\infty} C_m(\theta) \int_{\text{range}} (\hat{A}_m, \mathcal{G}) \quad \text{inverts } \mathbb{L}$$

Free Bulk Field

$$\phi, \Delta_d = \frac{d-2}{2}$$

Finite # of blocks on both sides.

$$C_\phi^1 f\left(\frac{d-2}{2}, \eta\right) + C_{\partial\phi}^1 f\left(\frac{d}{2}, \eta\right) = \left(\frac{\sin \theta}{2\sin \frac{d-2}{2}\theta}\right)^\Delta \left(C_\phi^1 f\left(\frac{d-2}{2}, \zeta\right) + C_{\partial\phi}^1 f\left(\frac{d}{2}, \zeta\right) \right)$$

Two free parameters:

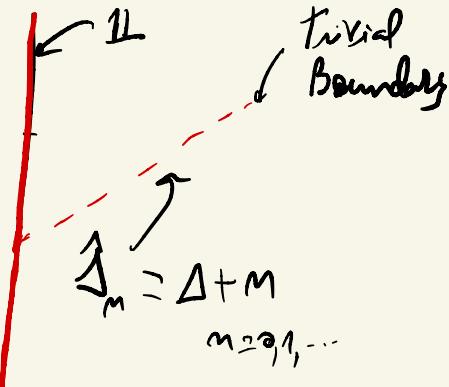
C_ϕ^1 (Neumann)

$C_{\partial\phi}^1$ (Dirichlet)

$$C_\phi^1 = \cos(\Delta_d \theta) C_\phi^1 + \frac{\sin(\Delta_d \theta)}{\Delta_d} C_{\partial\phi}^1$$

$$C_{\partial\phi}^1 = \Delta_d \sin(\Delta_d \theta) C_\phi^1 - \cos(\Delta_d \theta) C_{\partial\phi}^1$$

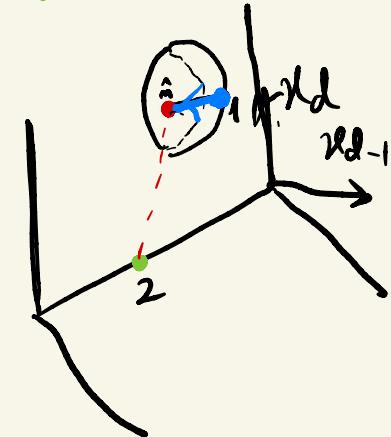
DD Vanishes unless $\theta = \Theta_d = \frac{2\pi}{d-2}$.



Generalization to Bulk - Edge 2-NT function

$$\langle \hat{O}_1 \hat{O}_2 \rangle$$

$$= \sum_m \frac{\mu^1_m}{(2\chi_{d-1})^{\Delta_1 - \tilde{\Delta}_m}} D(\alpha_{d-1}, \rho) \langle \hat{O}_m \hat{O}_2 \rangle$$



$$= \frac{1}{(2\chi_{d-1})^{\Delta_1 - \tilde{\Delta}_2}} g^{2\tilde{\Delta}_2} \sum_m C_m^{1,2} f_{\text{wall}}(\tilde{\Delta}_m, \tilde{\Delta}_2, \eta)$$

$$C_m^{1,2} = \mu_1^1 \mu_2^2$$

$$f_{\text{wall}}(\tilde{\Delta}_m, \tilde{\Delta}_2, \eta) = \eta^{\tilde{\Delta}_m - \tilde{\Delta}_2} F\left(\frac{\tilde{\Delta}_m - \tilde{\Delta}_2}{2}, \frac{1 + \tilde{\Delta}_m + \tilde{\Delta}_2}{2}, \frac{3}{2} - \frac{d}{2} + \tilde{\Delta}_m; -\eta\right)$$

Grossing:

$$\sum_m C_m f(\tilde{\Delta}_1, \tilde{\Delta}_2, \eta) = \left(\frac{\sin \phi}{\sin(\phi-\eta)}\right)^{\tilde{\Delta}_1 - \tilde{\Delta}_2} \sum_m C_m f(\tilde{\Delta}_m, \tilde{\Delta}_2, \eta)$$

Solutions to Crossing (Bulk-Edge 2-pt)

- Trivial Boundary ✓
- Free Bulk Field Same blocks as 1-pt.

$$C_{\phi}^{\dagger} = C_0((\Delta_d - \hat{\Delta}_2)\theta) C_{\phi}^{\dagger} + \frac{\sin((\Delta_d - \hat{\Delta}_2)\theta)}{\Delta_d - \hat{\Delta}_2} C_{\partial_2 \phi}^{\dagger}$$

$$C_{\partial_2 \phi}^{\dagger} = (\Delta_d - \hat{\Delta}_2) \sin((\Delta_d - \hat{\Delta}_2)\theta) C_{\phi}^{\dagger} - C_0((\Delta_d - \hat{\Delta}_2)\theta) C_{\partial_2 \phi}^{\dagger}$$

Can now solve for DD at arbitrary θ , If:

$$\hat{\Delta}_2 = \frac{d}{2} - 1 + n \frac{\pi}{\theta}$$

$$\langle O \hat{O}_{n=1} \rangle = \frac{\sin(\frac{\pi \phi}{\theta})}{r^{\Delta_d + \hat{\Delta}_2}}$$

- Can find GFF solution ($\Delta_d \rightarrow \Delta$),
With ∞ -many operators on both sides.

Summary

- Critical systems near an Edge have interesting Observables
- We can use Conformal invariance and try to bootstrap them

Future Directions

- $O(\epsilon)$ Bootstrap: Already need ∞ -many blocks.
- Numerical Bootstrap in 3d (or other, 4- ϵ)
 - Glisoggi (Sims 4 cells are not sign-definite)
 - SDP assuming positivity?
- Interactions of different co-dimension?
- Applications to Entanglement entropy?

THANK YOU!

Details on Cardy's Paper

$$g_0(\vec{r}, r, \phi, r', \phi')$$

$$= \frac{2}{\theta} \sum_{m=1}^{\infty} \int_0^{\infty} \mu d\mu \frac{J_{\frac{m\pi}{\theta}}(\mu r) J_{\frac{m\pi}{\theta}}(\mu r')}{(k^2 + m^2) + \mu^2} \sin\left(\frac{m\pi}{\theta}\phi\right) \sin\left(\frac{m\pi}{\theta}\phi'\right)$$

$$= \frac{2}{\theta} \sum_{m=1}^{\infty} I_{\frac{m\pi}{\theta}}(kr_{<}) I_{\frac{m\pi}{\theta}}(kr_{>}) \sin\left(\frac{m\pi}{\theta}\phi\right) \sin\left(\frac{m\pi}{\theta}\phi'\right)$$

$$r_{<} = \min(r, r') \quad r_{>} = \max(r, r')$$

For general C

$$\Psi(\vec{r}, r, \phi) = \int \frac{d^{d-2} \vec{k}}{(2\pi)^{d-2}} e^{i \vec{k} \cdot \vec{r}} \sum_{m=0}^{\infty} I_{\frac{m\pi}{\theta}}(kr) \left[A^{(m)}(\vec{k}) \sin\left(\frac{m\pi}{\theta}\phi\right) + B^{(m)}(\vec{k}) \cos\left(\frac{m\pi}{\theta}\phi\right) \right]$$

($\rightarrow \infty$ $B^{(n)} \rightarrow 0$; leading term is $A^{(1)}$ as $r \rightarrow 0$)

Bulk Renormalization

$$\begin{aligned}\psi &= Z_b^{1/2} \psi_R \quad \left. \begin{array}{l} \text{Critical} \\ \text{Mass Renormalization} \end{array} \right\} \\ g &= k^\epsilon Z_m \mu_0 \quad Z_\phi^2\end{aligned}$$

Without edge fields, Majorana conductors form
(Rb / Callan - Symanzik equation)

Adding Edge renormalization:

$$H + h_2 \int_{\vec{E}} d\vec{n} A^{(1)}(\vec{n}) \rightarrow Z_{h_2}$$

Bulk - Edge Rb equation

$$\frac{\langle A^{(1)}(\vec{n}) A^{(1)}(\vec{n}') \rangle_c}{|n - \vec{n}'|} \propto \frac{1}{|n - \vec{n}'|^{d-2 + \gamma_{E,E}}}$$

$$\gamma_{E,E} = d-2-\zeta_E$$

$$\frac{\langle A^{(1)}(\vec{n}) \psi(\vec{n}, n, \phi) \rangle}{n} \propto \frac{1}{n^{d-2 + \gamma_{V,E}}}$$

$$\gamma_{V,E} = \frac{1}{2}(d-2) + \frac{1}{2}\gamma_{V,V} - \zeta_E$$

$$\frac{\langle A^{(1)}(\vec{n}) \psi(\vec{n}, n, 0) \rangle}{\gamma_{S,E}} \propto \frac{1}{n^{d-2 + \gamma_{S,E}}}$$

→ All Critical exponents and scaling relations

from $\gamma_{V,S,E}$ and v ($2 - \frac{1}{v} = -\kappa \frac{2 \ln 2 \kappa^2}{2 \kappa}$)

Can compute ζ_E at one-loop from R6 equation
of $\langle A^{(1)}(\vec{n}) A^{(1)}(\vec{n}') \rangle$ (Subtract poly in ϵ)

Details on Bootstrap paper

Bulk - Edge 2-PT Function

$$\langle O_1(\vec{r}, \chi_{d-1}, \chi_d) \hat{O}_2(\vec{r}', 0, 0) \rangle = \frac{g(\phi, \theta)}{r^2 (\Delta_2)_{(2\chi_{d-1})}^{\Delta_1 - \Delta_2}}$$

By Translation + SCT (can set $\vec{r}' = \vec{r} = 0$)

Edge
Scaling
Dimension

MC Almty-Osborn showed that:

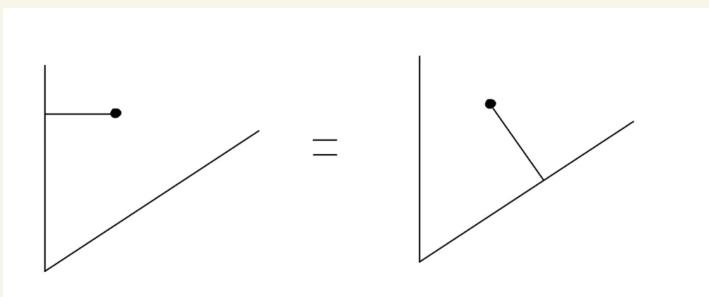
$$D[\chi_{d-1}, \partial_x^2, \partial_{x_d}] = \sum_{m=0}^{\infty} \frac{1}{m! (\Delta_{d-1} + \frac{3}{2} - \frac{d}{2})_m} \left(-\frac{1}{4} \chi_{d-1}^2 \left(\nabla^2 \partial_{x_d}^2 \right)^m \right)$$

$$f_{\text{wall}}(\Delta_m, \eta) = \eta^{\Delta_m} {}_2F_1\left(\frac{\Delta_m}{2}, \frac{\Delta_m + 1}{2}; \frac{3-d}{2} + \Delta_m; -\eta^2\right)$$

BOE limit $\chi_{d-1} \rightarrow 0 \rightarrow f \sim \eta^{\Delta_m} \checkmark$

Can also be obtained with **Belyaev-Gasin** eq. $\tilde{L}_g^2 = L_g^2$

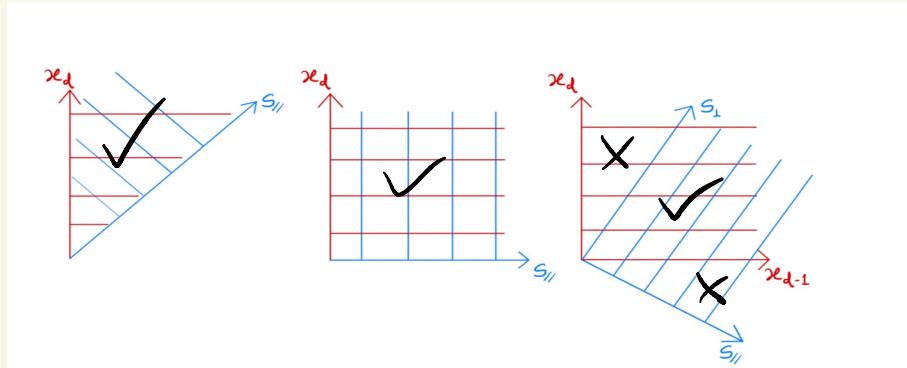
Crossing Equation and BOE Convergence



An illustrative case is $\Theta = \frac{\pi}{2}$. (η vs $\frac{1}{\eta}$)

$$\begin{aligned} f_{\text{small}}(\eta, \tilde{x}_1) &\xrightarrow{\eta \rightarrow 0} \eta \tilde{x}_1 + O(\eta^2) \\ f_{\text{large}}(\tilde{x}_1, \eta) &\xrightarrow{\eta \rightarrow 0} b_0 + b_1 \eta + \dots \end{aligned} \quad \begin{array}{l} \text{Different} \\ \text{asymptotic} \\ \text{behaviors} \\ \Downarrow \\ \text{Use Analytic} \\ \text{Bootstrap Mechanisms} \\ (\infty \text{ operators...}) \end{array}$$

BOE Convergence:



\Rightarrow Careful about $\Theta \rightarrow \pi$ limits!

$$f_{\text{wall}}(\hat{\Delta}_m, \hat{\Delta}_2, \xi) = \gamma^{\hat{\Delta}_m - \hat{\Delta}_2} F\left(\frac{\hat{\Delta}_m - \hat{\Delta}_2}{2}, \frac{1 + \hat{\Delta}_m + \hat{\Delta}_2}{2}, \frac{3}{2}, \frac{d}{2} + \hat{\Delta}_m; -\gamma^2\right)$$

Can also get for Cosine and Recover 1-pi as $\hat{\Delta}_2 = 0$

$$\boxed{\hat{\Delta}_2 = \frac{d}{2} - 1 + n \frac{\pi}{\theta}}$$

Similar Story for DN, NN.

$$\begin{matrix} d & \\ \downarrow & \downarrow \\ \frac{+\pi}{2\theta} & \frac{+2\pi}{\theta} \end{matrix}$$