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A few steps towards the numerical evaluation of multivariable hypergeometric functions

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Table of Contents

1. Motivations
2. A brief presentation of multivariable hypergeometric series
3. Mellin-Barnes integrals and hypergeometric functions
4. A digression on conformal Feynman integrals
5. Back to hypergeometric functions and their linear transformations
6. Conclusion and Outlook

Table of Contents

1. Motivations
2. A brief presentation of multivariable hypergeometric series
3. Mellin-Barnes integrals and hypergeometric functions
4. A digression on conformal Feynman integrals
5. Back to hypergeometric functions and their linear transformations
6. Conclusion and Outlook

Motivations

- Multivariable hypergeometric functions (MHF) are ubiquitous in physics.
- Feynman integrals can be written in terms of (combinations of) MHF.
- Well-known mathematical softwares such as *Mathematica* or *Maple* do not provide the numerical evaluation of MHF in their kernel.
- A recent and important result in the theory of Mellin-Barnes integrals has naturally driven us to this field of investigations.

  [Ananthanarayan, Banik, SF, Ghosh, Phys.Rev.Lett. 127 (2021) 15, 151601]

Table of Contents

1. Motivations
2. A brief presentation of multivariable hypergeometric series
3. Mellin-Barnes integrals and hypergeometric functions
4. A digression on conformal Feynman integrals
5. Back to hypergeometric functions and their linear transformations
6. Conclusion and Outlook

Single variable hypergeometric series

Gauss ${}_2F_1$ hypergeometric series:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \qquad |z|<1 $$

Generalized ${}_pF_q$ hypergeometric series:

$$ {}_pF_q((a_i);(b_j);z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} $$

where $(a)_n \doteq \Gamma(a+n)/\Gamma(a)$ (Pochhammer symbol).

Fox ${}_p\Psi_q$ hypergeometric series:

$$ {}_p\Psi_q((a_i,A_i);(b_j,B_j);z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\prod_{i=1}^{p}(a_i)_{A_i n}}{\prod_{j=1}^{q}(b_j)_{B_j n}} \qquad 1+\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i \geq 0 $$

Double variable hypergeometric series (of order 2)

There are 34 distinct series of order 2.

Appel series:

- 4 series F_1, F_2, F_3 and F_4.

Example:

$$F_1(a; b, b'; c; x, y) = \sum_{m,n=0}^{\infty} \frac{x^m}{m!} \frac{y^n}{n!} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \qquad |x| < 1, |y| < 1$$

- 7 confluent forms: $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1$ and Ξ_2.

Example:

$$\Phi_1(a; b; c; x, y) = \sum_{m,n=0}^{\infty} \frac{x^m}{m!} \frac{y^n}{n!} \frac{(a)_{m+n}(b)_m}{(c)_{m+n}} \qquad |x| < 1, |y| < \infty$$

$$\Phi_1(a; b; c; x, y) = \lim_{|b'|\to\infty} F_1(a; b, b'; c; x, y/b')$$

Double variable hypergeometric series (of order 2)

Horn series:

- 10 series G_1, G_2, G_3 and $H_1, ..., H_7$.
  Example:

$$H_1(a, b, c; d; z) = \sum_{m,n=0}^{\infty} \frac{x^m}{m!} \frac{y^n}{n!} \frac{(a)_{m-n}(b)_{m+n}(c)_n}{(d)_m}$$

$$|y| + 2\sqrt{|xy|} < 1$$

- 13 confluent forms: $\Gamma_1, \Gamma_2, \mathrm{H}_1, ..., \mathrm{H}_{11}$.

Double variable hypergeometric series (higher order)

Kampé de Fériet series:

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix} a_1, \ldots, a_p \,:\, b_1, \ldots, b_q\,;\, c_1, \ldots, c_k \\ \alpha_1, \ldots, \alpha_l \,:\, \beta_1, \ldots, \beta_m; \gamma_1, \ldots, \gamma_n \end{matrix}\,\middle|\, x, y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{x^m}{m!}\frac{y^m}{m!}\frac{\prod_{i=1}^{p}(a_i)_{r+s}\prod_{i=1}^{q}(b_j)_r\prod_{i=1}^{k}(c_i)_s}{\prod_{i=1}^{l}(\alpha_i)_{r+s}\prod_{i=1}^{m}(\beta_i)_r\prod_{i=1}^{n}(\gamma_i)_s}$$

Triple hypergeometric series (of order 2)

Srivastava and Karlsson (1985) have listed in their book the 205 triple hypergeometric series of order 2, among which:

- Srivastava $F^{(3)}$ series which unifies the 14 Lauricella series (F_A, F_B, F_C and F_D are the most known) and the 3 Srivastava series (H_A, H_B and H_C).
- Pandey G_A and G_B series and Srivastava G_C series.
- etc.

Convergence regions are also known.
This is the highest dimensional exhaustive list of series of order 2:
$1 \to 34 \to 205 \to ?$

Multivariable hypergeometric series

Further generalizations have been studied.

Example: Srivastava-Daoust generalized Lauricella series:

$$F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}[x_1,\dots,x_n]$$
$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{{x_1}^{m_1}}{m_1!}\cdots\frac{{x_n}^{m_n}}{m_n!}\frac{\prod_{i=1}^{A}(a_i)_{m_1\theta'_i+\dots+m_n\theta_i^{(n)}}\prod_{i=1}^{B'}(b'_i)_{m_1\phi'_i}\cdots\prod_{i=1}^{B^{(n)}}(b_i^{(n)})_{m_n\phi_i^{(n)}}}{\prod_{i=1}^{C}(c_i)_{m_1\psi'_i+\dots+m_n\psi_i^{(n)}}\prod_{i=1}^{D'}(d'_i)_{m_1\delta'_i}\cdots\prod_{i=1}^{D^{(n)}}(d_i^{(n)})_{m_n\delta_i^{(n)}}}$$

where $\theta_i^{(k)},\phi_i^{(k)},\psi_i^{(k)}$ and $\delta_i^{(k)}$ are positive numbers.

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Multivariable hypergeometric series

All these multivariable hypergeometric series correspond to some functions which one would like to numerically evaluate outside of the regions of convergence of the former.

Only the Appell F_1 function has been implemented in the kernel of *Mathematica* !

Only the four Appell functions are in-built functions of *Maple* (since 2017...) !

Moreover, these implementations need improvement/checks.

Therefore, a lot of work remains to be done... but how can one perform the numerical evaluation of these hypergeometric functions?

Table of Contents

1. Motivations
2. A brief presentation of multivariable hypergeometric series
3. Mellin-Barnes integrals and hypergeometric functions
4. A digression on conformal Feynman integrals
5. Back to hypergeometric functions and their linear transformations
6. Conclusion and Outlook

MB integrals and hypergeometric functions

Mellin-Barnes representations of hypergeometric functions can be used to analytically continue their series representations and to study their transformation theory.

Example 1: Gauss ${}_2F_1$ hypergeometric function

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{ds}{2i\pi} (-z)^s \Gamma(-s) \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} $$

Transformation formula (analytic continuation):

$$ \begin{aligned} {}_2F_1(a,b;c;z) = & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a}{}_2F_1(a,a-c+1;a-b+1;1/z) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b}{}_2F_1(b,b-c+1;b-a+1;1/z) \end{aligned} $$

MB integrals and hypergeometric functions

Example 2: Srivastava's H_C triple hypergeometric function (related to the massive conformal 3-point Feynman integral [Loebbert, Miczajka, Müller, Münster, Phys.Rev.Lett. 125 (2020) 9, 091602]):

- Series representation: $H_C(a,b,c;d;x,y,z) = \frac{\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(c)}$
$$\times \sum_{m,n,p=0}^{\infty} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{\Gamma(a+m+n)\Gamma(b+m+p)\Gamma(c+n+p)}{\Gamma(d+m+n+p)}$$
- MB representation:
$$H_C(a,b,c;d;x,y,z) = \frac{\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(c)}$$
$$\times \int_{-i\infty}^{i\infty}\frac{dz_1}{2i\pi}\int_{-i\infty}^{i\infty}\frac{dz_2}{2i\pi}\int_{-i\infty}^{i\infty}\frac{dz_3}{2i\pi}(-x)^{z_1}(-y)^{z_2}(-z)^{z_3}$$
$$\times \Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\frac{\Gamma(a+z_1+z_2)\Gamma(b+z_1+z_3)\Gamma(c+z_2+z_3)}{\Gamma(d+z_1+z_2+z_3)}$$

MB integrals and hypergeometric functions

13 different linear transformation formulas [Ananthanarayan, Banik, SF, Ghosh, Phys.Rev.D 103 (2021) 9, 096008] can be extracted from this 3-fold MB representation, among which one well-known result:

$$
\begin{aligned}
&H_C(a,b,c;d;x,y,z)\\
&=\frac{\Gamma(d)\Gamma(c-b)}{\Gamma(d-b)\Gamma(c)}(-z)^{-b}G_C(b,a,b-d+1;b-c+1;x/z,1/z,y)\\
&+\frac{\Gamma(d)\Gamma(b-c)}{\Gamma(d-c)\Gamma(b)}(-z)^{-c}G_C(c,a,c-d+1;c-b+1;y/z,1/z,x)
\end{aligned}
$$

where

$$
G_C(a,b,c;d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n}(b)_{m+p}(c)_{n-p}}{(d)_{m+n-p}}\frac{x^m y^n z^p}{m!\,n!\,p!}
$$

How did we get these linear transformations?

An important progress has been recently achieved in the theory of multifold MB integrals [Ananthanarayan, Banik, SF, Ghosh, Phys.Rev.Lett. 127 (2021) 15, 151601]:

- We have found the first technique to compute analytically N-fold MB integrals of arbitrary complexity.
- This computational method is based on a geometrical approach using conic hulls.

The Conic Hull Method

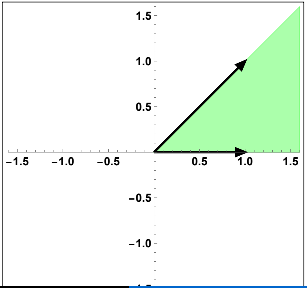
What is a conic hull?

- Conic hulls are semi-infinite geometric regions.

$$\{p + s_1 v_1 + \cdots + s_n v_n | s_i \in \mathbb{R}_+\}$$

where, the point p is the vertex and v_i's are the basis vectors.

- For example, if $p = (0, 0)$ and $v_1 = (1, 0)$, $v_2 = (1, 1)$



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Conic Hull Approach
The Method

Brief Overview for a given N-fold MB integrals: (Non-Resonant)

- **Step 1:** Find all possible N-combinations of numerator gamma functions and retain non-singular ones.
- **Step 2:** Associate a series (building block) with each combination.
- **Step 3:** Construct a conic hull for each combination/series.
- **Step 4:** Largest intersecting subsets of conic hulls give series representations of the MB integral.
- **Step 5:** The intersecting region gives the master conic hull.

This algorithm has been implemented in the *Mathematica* package MBConicHulls.wl [Ananthanarayan, Banik, SF, Ghosh, Phys.Rev.Lett. 127 (2021) 15, 151601]

Comparison with other approaches
(in the context of Feynman integrals calculations)

In comparison with the Yangian Bootstrap approach, the Method of Brackets, the Negative Dimension approach, etc., our method:

- bypasses the convergence analysis.
- Derives overall constant factors of building blocks analytically.
- Can be applied to both arbitrary (non-resonant) and unit (resonant) propagator power cases straightforwardly.

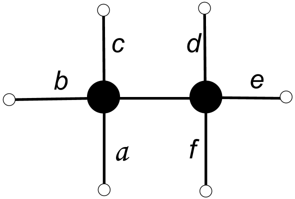
Table of Contents

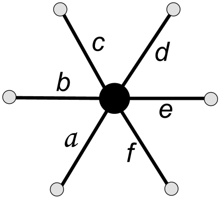
1. Motivations
2. A brief presentation of multivariable hypergeometric series
3. Mellin-Barnes integrals and hypergeometric functions
4. A digression on conformal Feynman integrals
5. Back to hypergeometric functions and their linear transformations
6. Conclusion and Outlook

A digression on conformal Feynman integrals

Hexagon and Double Box

We computed the previously unsolved dual-conformal fishnet Double-Box and Hexagon diagrams [Ananthanarayan, Banik, SF, Ghosh, Phys.Rev.D 102 (2020) 9, 091901 (Rapid Communication)]:





both of which have a 9-fold MB representation.

A digression on conformal Feynman integrals

Hexagon and Double Box

The MB representation of the Hexagon is [Loebbert, Müller, Münkler, Phys.Rev.D 101 (2020) 6, 066006]

$$\frac{1}{(2\pi i)^9}\int_{-i\infty}^{+i\infty}\dots\int_{-i\infty}^{+i\infty}\prod_{i=1}^{9}\left(dz_i\, w_i^{z_i}\Gamma(-z_i)\right)$$
$$\times\,\Gamma(\alpha_1+z_1+\dots+z_9)\Gamma(\alpha_2+z_1+z_5+z_8+z_9)$$
$$\times\,\Gamma(\alpha_3+z_2+z_6+z_7+z_8)\Gamma(\alpha_4+z_3+\dots+z_6)$$
$$\times\,\Gamma(\alpha_5-z_4-\dots-z_9)\Gamma(\alpha_6-z_1-z_2-z_3-z_5-z_6-z_8)$$

The α_i are linear combinations of propagator powers.

A digression on conformal Feynman integrals

Hexagon and Double Box

- The total number of building blocks for the Double-Box and Hexagon are 4834 and 2530, respectively.
- We solved both as a linear combination of 44 and 26 building blocks, respectively [Ananthanarayan, Banik, SF, Ghosh, Phys.Rev.D 102 (2020) 9, 091901 (Rapid Communication)].
- We also solved them for the highly non-trivial unit propagator case [Ananthanarayan, Banik, SF, Ghosh, Phys.Rev.Lett. 127 (2021) 15, 151601].

A digression on conformal Feynman integrals
Hexagon and Double Box

Numerical Comparison for Hexagon		
Upper Sum Limit	Series Representation (Time)	Feynman Parametrization (Time)
3	636.76884 (3 min)	636.76882 (9 hours)

Table: Computed for $u_1 = 1, u_2 = 10^{12}, u_3 = 1/10^{12}, u_4 = 1, u_5 = 1, u_6 = 100, u_7 = 1/100, u_8 = 10000, u_9 = 1/10^8$ for propagator powers $a = 42/100, b = 11/100, c = 15/100, d = 32/100, e = 59/100, f = 55/100$

Table of Contents

1. Motivations
2. A brief presentation of multivariable hypergeometric series
3. Mellin-Barnes integrals and hypergeometric functions
4. A digression on conformal Feynman integrals
5. Back to hypergeometric functions and their linear transformations
6. Conclusion and Outlook

Back to hypergeometric functions and their linear transformations

Using the transformation theory of Gauss ${}_2F_1$ (and other results such as Carlson's Appell F_1 identity) at the series (or integral) level one can obtain many transformation formulas for a large class of multivariable hypergeometric series (functions).

- These transformation formulas (analytic continuations, etc.) can be used in their series form to compute the hypergeometric functions outside of the region of convergence of their usual (defining) series representations.
- As we have seen in the example of the hexagon, series can be very efficient for numerical purpose.

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Back to hypergeometric functions and their linear transformations

Examples:

- At the series level: starting from Appell F_2 series, we derived 44 different linear transformations which form the basis of the *Mathematica* package AppellF2.wl for its numerical evaluation [Ananthanarayan, Bera, SF, Marichev, Pathak, e-Print: 2111.05798 [math.CA] (2021)]. This computational approach of linear transformations at the series level has also been automatized in the *Mathematica* package Olsson.wl [Ananthanarayan, Bera, SF, Pathak, e-Print: 2201.01189 [cs.MS] (2021)].
- At the integral level: for Srivastava H_C function, more than 50 different MB representations can be obtained → hundreds of linear transformations. [SF, Suchet-Bernard, e-Print: 2205.06247 [math-ph] (2022)].

Table of Contents

1. Motivations
2. A brief presentation of multivariable hypergeometric series
3. Mellin-Barnes integrals and hypergeometric functions
4. A digression on conformal Feynman integrals
5. Back to hypergeometric functions and their linear transformations
6. Conclusion and Outlook

Conclusion
Summary

- We have quickly presented the bestiary of hypergeometric functions.
- From the point of view of their numerical evaluation, starting at 2 variables, everything has to be done.
- Two different techniques for the derivation of their transformation theory have been implemented in *Mathematica* packages (Olsson.wl and MBConicHulls.wl).
- The second approach uses N-fold MB integrals that one can now compute analytically for arbitrary positive N thanks to a simple geometrical method based on conic hulls.
- This technique allows faster numerical computation and gives smaller numbers of series than MBsums.m.

Conclusion
Future Work

- We have shown how this method can also solve complicated conformal Feynman integrals. The case of phenomenological Feynman integrals now has to be considered.
- Solve the complex case for numerical evaluation of MHF.
- Build the package for Horn and other Appell functions from AppellF2.wl.
- Build a package for the application of Horn's theorem of convergence for higher dimensional hypergeometric series.
- Improve the speed of MBConicHulls.wl.